Staple!

Team W____

Number Theory 2
MAT4930 5662

Home-W

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Touch: 2Jul2018

Hello. Essays violate the CHECKLIST at *Grade Peril*! Exam is due **5PM, Tuesday, 10Oct2006**, slid under LIT402. Please email me afterwards.

W1: Short answer: Show no work. Write **DNE** in a blank if the described object does not exist or if the indicated operation cannot be performed.

___ ___ a The unique integer $k \in (-90 \,..\, 90]$ with

$$k \equiv_4 27\,,\quad k \equiv_9 19\,,\quad k \equiv_5 8\,,\quad \text{is } k = \underline{\qquad\qquad}.$$

An explicit ring-iso $\mathbb{Z}_4 \times \mathbb{Z}_9 \times \mathbb{Z}_5 \hookrightarrow \mathbb{Z}_{180}$ is

$$(x, y, z) \longmapsto \langle 45Ax + 20By + 36Cz \rangle_{180}\,, \quad \text{where}$$

$A=$ ______ $\in [0 \,..\, 4)$, $B=$ ______ $\in [0 \,..\, 9)$, $C=$ ______ $\in [0 \,..\, 5)$.

___ ___ b The last two digits of $K := 19^{7^{21^{57}}}$ are ___ ___.

___ ___ c Writing 100! in decimal, it ends (on the right) with ______ zeros.

___ ___ d $\sum_{n=1}^{\infty} \mu(n!) =$ ______________.

Essay questions: Write in complete sentences and also fill-in the blanks. Each essay starts a new page.

W2: Consider the inequalities

$$\frac{B}{\sqrt{\frac{1}{2}+n}} \cdot 4^n \overset{1*}{\geqslant} \binom{2n}{n} \overset{2*}{\geqslant} \frac{A}{\sqrt{n}} \cdot 4^n\,.$$

Setting $B := 1$ and $A := \frac{1}{2}$, give induction proofs that $(1*)$ and $(2*)$ hold for all posints n. (Can you improve the result?)

W3: [Möbius] Let $\text{cis}(\theta)$ be $\cos(\theta) + \boldsymbol{i}\sin(\theta)$, the complex number $\mathrm{e}^{\boldsymbol{i}\theta}$. In $\mathbb{C}$, the N^{th}-roots of unity are $\omega_j := \text{cis}(j \cdot \frac{2\pi}{N})$, for $j = 1, 2, \ldots, N$.

An N^{th}-root ω is ***primitive*** if, for all $k < N$, this ω is *not* a k^{th}-root of unity. Use V_N for the set of primitive N^{th}-roots. i Prove that $|V_N| = \varphi(N)$.

ii Define a complex poly

$$\mathbf{C}_N(z) := \prod_{\omega \in V_N} [z - \omega]\,.$$

Prove that each coeff of $\mathbf{C}_N$ is real. [*Eg:* $\mathbf{C}_1(z) = z - 1$. $\mathbf{C}_2(z) = z + 1$. $\mathbf{C}_3(z) = z^2 + z + 1$.]

iii Let $F_N(z) := z^N - 1$. Prove $F_N = \prod_{d \bullet| N} \mathbf{C}_d$, where the product is taken over all positive divisors of N. Find a way to use the *Möbius inversion formula* to write

$$\dagger\!: \qquad \mathbf{C}_N(z) = \prod_{d \bullet| N} \begin{bmatrix}\text{Something involving} \\ N, d \text{ and } F\end{bmatrix}$$

in terms of the Möbius μ fnc. Use this to show that $\mathbf{C}_N$ is an *intpoly*!

Use your formula to write $\mathbf{C}_{24}(z)$ in std. form. (What degree do we expect it to have?)

For posint L, write $\mathbf{C}_{3^L}(z)$ in std. form.

W4: Let p_n denote the n^{th}-prime. Use Chebyshev's Thm (5.1 on P.75 of Shoup) to prove that there exists a constant $K \geqslant 1$ so that

$$\forall n \in [2 \,..\, \infty)\,: \qquad p_n \leqslant K \cdot [n \cdot \log(n)]\,.$$

[*Hint:* What is $\pi(p_n)$?] Can you prove a stronger result? (We haven't proved PNT, so we don't have that.)

End of Home-W

W1:	___ ___	80pts
W2:	___ ___	85pts
W3:	___ ___	85pts
W4:	___ ___	85pts
Total:	___ ___ ___	335pts

HONOR CODE: *"I have neither requested nor received help on this exam other than from my team-mates and my professor (or his colleague)."* Name/Signature/Ord

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