Sets and Logic MHF3202 8768 | **Home-W** | Prof. JLF King 21Feb2017

Due **BoC, Monday, 27Feb2017**, wATMP! Write **DNE** in a blank if the described object does not exist or if the indicated operation cannot be performed.

W1: *Show no work.*

___ ___ a) The ***Threeish-numbers*** comprise $\mathcal{T} := 1 + 3\mathbb{N}$. $\mathcal{T}$-number $385 \stackrel{\text{note}}{=\!=} 35 \cdot 11$ is $\mathcal{T}$*-irreducible*: T F

Threeish $N := 85$ is **not** $\mathcal{T}$*-prime* because $\mathcal{T}$-numbers $J :=$ ________ and $K :=$ ________ satisfy that $N \bullet\!| \, [J \cdot K]$, **yet** $N \nmid J$ and $N \nmid K$.

Also, $\mathcal{T}$-GCD$(175, 70) =$ ________.

___ ___ b) On $\Omega := [1..29] \times [1..29]$, define binary-relation $\mathbf{C}$ by: $(x, \alpha)\, \mathbf{C}\, (y, \beta)$ IFF $x \cdot \beta \equiv_{30} y \cdot \alpha$. Statement "*Relation* $\mathbf{C}$ *is an **equivalence relation***" is: T F

___ ___ c) On $\mathbb{Z}_+$, write $x \$ y$ IFF $\text{GCD}(x, y) \geqslant 2$. So \$ is

Circle

Transitive: T F. **Symm.**: T F. **Reflex.**: T F.

On $\mathbb{Z}$, say that $x \nabla y$ IFF $x - y < 1$. Then ∇ is:

Trans.: T F. **Symm.**: T F. **Reflex.**: T F.

___ ___ d) On a K-element set, the number of reflexive symmetric binrels is ________.

On a 4-set, there are ________ many equiv.relations.

Carefully TYPE *your essays, double–spaced. I suggest* LaTeX*, but other systems are ok too.*

W2: In our Velleman text, solve problem #**12**$^{P.}$*277*. Let $\mathbf{E}_n$ be the equilateral triangle with side-length 2^n. This $\mathbf{E}_n$ can be tiled in an obvious way by 4^n many little-triangles [copies of $\mathbf{E}_0$]; see picture P.277. The "***punctured*** $\mathbf{E}_n$", written $\widetilde{\mathbf{E}_n}$, has its topmost copy of $\mathbf{E}_0$ removed.

A ***(trape)zoid***, $\mathbf{T}$, comprises three copies of $\mathbf{E}_0$ glued together in a row, rightside-up, upside-down, rightside-up [picture P.277]. [A ***zoid-tiling*** allows all three rotations of $\mathbf{T}$.]

i) PROVE: *For each n, board $\widetilde{\mathbf{E}_n}$ admits a zoid-tiling.*

ii) Let Δ_k be the equilateral triangle of sidelength k; so $\mathbf{E}_n$ is Δ_{2^n}. Triangle Δ_k comprises k^2 little-triangles.

For what values of k does Δ_k admit a zoid-tiling?

For which k does $\widetilde{\Delta_k}$ admit a zoid-tiling?

iii) An ***Lmino*** (pron. "ell-mino") comprises three  squares in an "L" shape (all four orientations are allowed).

Let $\mathbf{S}_n$ be the $2^n \times 2^n$ square board, comprising 4^n ***squaries*** (little squares). Have $\widetilde{\mathbf{S}_n}$ be the board with one corner squarie removed. Velleman inductively shows, pp.272-275, that each $\widetilde{\mathbf{S}_n}$ is Lmino-tilable (by $[4^n - 1]/3$ Lminos, of course). Further, with $\mathbf{S}'_n$ denoting $\mathbf{S}_n$ with an *arbitrary* puncture, V. proves that every $\mathbf{S}'_n$ is Lmino-tilable.

Generalize this to three-dimensions. Let $\mathbf{C}_n$ denote the $2^n \times 2^n \times 2^n$ cube, $\widetilde{\mathbf{C}_n}$ the corner-punctured cube, and let $\mathbf{C}'_n$ be $\mathbf{C}_n$ but with an arbitrary ***cubie*** removed.

What is the 3-dimensional analog of an Lmino? Calling it a "**3-*mino***", how many cubies does it have? [Provide a drawing of your 3-mino.] PROVE: *Every $\mathbf{C}'_n$ admits a 3-mino–tiling.* [Provide also pictures showing your ideas.]

iv) Generalize to K-dim(ensional) space, with $\mathbf{C}_{n,K}$ being the $2^n \times \overset{K}{\cdots} \times 2^n$ cube, having $[2^n]^K = 2^{nK}$ many K-dim'al cubies. As before, let $\mathbf{C}'_{n,K}$ be $\mathbf{C}_{n,K}$ with an arbitrary cubie removed.

*What is your K-**mino** with which you will tile, and how many cubies does it have?* (So a 2-mino is our Lmino.) PROVE: *Every $\mathbf{C}'_{n,K}$ admits a K-mino–tiling.*

W3: [A dodecahedron is a regular polyhedron having 12 faces, 20 vertices and 30 edges; the faces are pentagons.] Two vertices of a regular dodecahedron are ***cousins*** if they are *distinct* vertices of a common face. [Each vertex has $[3 \cdot 4] - 3 = 9$ cousins.] Write $v \sim w$ to indicate that v and w are cousins. Easily, $\sim$ is symmetric, and anti-reflexive. You can check that $\sim$ is not transitive.

A ***labeling*** of a regular dodecahedron assigns, to each vertex, a *positive integer*. A labeling is ***legal*** IFF *no* pair $v \sim w$ of vertices is assigned the same label.

i) Prove there is no legal labeling with vertex-sum [the sum of the 20 labels] equaling 59.

ii) Let $\mathcal{S} \subset \mathbb{Z}_+$ be the *set* of vertex-sums obtainable from legal-labelings. Characterize $\mathcal{S}$ explicitly, with proof. You will likely need to construct some particular legal-labelings. [You showed, above, that $\mathcal{S} \not\ni 59$.]

W1:	___ ___ ___	125pts
W2:	___ ___ ___	130pts
W3:	___ ___	95pts
Total:	___ ___ ___	350pts

HONOR CODE: *"I have neither requested nor received help on this exam other than from my team-mates and my professor (or his colleague)."* *Name/Signature/Ord*

Ord:

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