Volume of the $\mathfrak{D}$-ball and the Gamma function

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ABSTRACT: (See "Joe's Circle Puzzle" for an application of this, in PROBLEMS NB)

We are born knowing that...

For $\mathfrak{D} \in \mathbb{Z}_+$, the $[\mathfrak{D}-1]$-***sphere*** of radius $r \geq 0$ is the set of tuples

¥: $$(x_1, x_2, \ldots, x_\mathfrak{D}) \in \mathbb{R}^{\times\mathfrak{D}}$$

with ${x_1}^2 + \cdots + {x_\mathfrak{D}}^2 = r^2$. In contrast, the [closed] $\mathfrak{D}$-***ball***, for $\mathfrak{D} \in \mathbb{N}$, comprises those (¥) with ${x_1}^2 + \cdots + {x_\mathfrak{D}}^2 \leq r^2$. So the $[\mathfrak{D}-1]$-sphere is the surface of the $\mathfrak{D}$-ball.

Examples. The 2-sphere is the usual sphere; the 1-sphere is the *circle*; the 0-sphere comprises the two points $\pm r$, which are the endpoints of the 1-*ball* $[-r, r]$, which is an *interval*.

The 2-ball is the *disk*, and the 3-ball is the usual (solid) ball, e.g a bowling ball. (The 0-ball is the one point of $\mathbb{R}^{\times 0}$; the empty tuple.) □

Prolegomena. Define constants $\beta_\mathfrak{D}, \sigma_\mathfrak{D}$, for $\mathfrak{D} = 0, 1, 2, \ldots$, from the ball of radius r:

$$\beta_\mathfrak{D} \cdot r^\mathfrak{D} := \mathfrak{D}\text{-volume of the } \mathfrak{D}\text{-ball};$$
$$\sigma_\mathfrak{D} \cdot r^\mathfrak{D} := \mathfrak{D}\text{-SurfaceArea of } [\mathfrak{D}+1]\text{-ball}$$

For a ball, surface area is the growth rate of volume and so $\sigma_{\mathfrak{D}-1} r^{\mathfrak{D}-1}$ equals

$$\lim_{\Delta r \searrow 0} \frac{\beta_\mathfrak{D}[r + \Delta r]^\mathfrak{D} - \beta_\mathfrak{D} r^\mathfrak{D}}{\Delta r} \stackrel{\text{note}}{=\!=} \frac{\mathrm{d}}{\mathrm{d}r}(\beta_\mathfrak{D} \cdot r^\mathfrak{D}).$$

Consequently

1: $$\sigma_{\mathfrak{D}-1} = \mathfrak{D} \cdot \beta_\mathfrak{D}, \quad \text{for } \mathfrak{D} \in \mathbb{Z}_+.$$

(Do we realize, for $\mathfrak{D} = 3, 2, 1$, that we already knew (1); we just didn't know that we knew it?)

Let $J := \int_{-\infty}^{+\infty} \mathrm{e}^{-x^2}\,\mathrm{d}x$. We use the ***PCT*** ("polar coordinate trick") to show that $J = \sqrt{\pi}$. We integrate the cartesian-square of the integrand to conclude that

2.1: $$\begin{aligned} J^2 &= \left[\int_{-\infty}^{+\infty} \mathrm{e}^{-[x^2]}\,\mathrm{d}x\right] \cdot \left[\int_{-\infty}^{+\infty} \mathrm{e}^{-[y^2]}\,\mathrm{d}y\right] \\ &= \int_{-\infty}^{+\infty} \mathrm{e}^{-[x^2+y^2]} \cdot \mathrm{d}(x,y) \\ &= \int_0^{+\infty} \mathrm{e}^{-r^2} \cdot \underbrace{2\pi r \cdot \mathrm{d}r}_{\text{Area of radius-}r\text{ annulus of thickness } \mathrm{d}r}. \end{aligned}$$

Hence $J^2 = \pi \cdot [-\mathrm{e}^{-r^2}]\Big|_{r=0}^{r=+\infty} = \pi$. Since J is *the integral of a non-negative fnc*, nec. $J \geq 0$. Thus

2.2: $$\int_{-\infty}^{+\infty} \mathrm{e}^{-x^2}\,\mathrm{d}x = \sqrt{\pi}.$$

A Device used Twice is a Technique.

For $\mathfrak{D}$ a posint, then, $\pi^{\mathfrak{D}/2}$ equals the product

$$\begin{aligned} [\pi^{\frac{1}{2}}]^\mathfrak{D} &= \left[\int_{-\infty}^{+\infty} \mathrm{e}^{-[x_1]^2}\,\mathrm{d}x_1\right] \cdot \ldots \cdot \left[\int_{-\infty}^{+\infty} \mathrm{e}^{-[x_\mathfrak{D}]^2}\,\mathrm{d}x_\mathfrak{D}\right] \\ &= \int_{-\infty}^{+\infty} \mathrm{e}^{-[{x_1}^2+\cdots+{x_\mathfrak{D}}^2]} \cdot \mathrm{d}(x_1, \ldots, x_\mathfrak{D}). \end{aligned}$$

Converting this last integral into polar coordinates yields

2.1b: $$\pi^{\mathfrak{D}/2} = \int_0^{+\infty} \mathrm{e}^{-r^2} \cdot \underbrace{\sigma_{\mathfrak{D}-1} r^{\mathfrak{D}-1} \cdot \mathrm{d}r}_{\text{Infinitesimal } \mathfrak{D}\text{-volume of thickened } [\mathfrak{D}-1]\text{-sphere.}}.$$

Here, $\sigma_{\mathfrak{D}-1} r^{\mathfrak{D}-1}$ is the $[\mathfrak{D}-1]$-area of the spherical shell of radius r. Multiplying by thickness $\mathrm{d}r$ gives $\sigma_{\mathfrak{D}-1} r^{\mathfrak{D}-1}\,\mathrm{d}r$ as the infinitesimal $\mathfrak{D}$-volume of the thickened sphere.

Let's explore this last integral.

The Gamma Function

For each $\mathfrak{D} \in \mathbb{R}$, let $\boxed{I_\mathfrak{D} := \int_0^{+\infty} \mathrm{e}^{-r^2} r^{\mathfrak{D}-1}\,\mathrm{d}r}$. For $\mathfrak{D} \leq 0$ this integral is $+\infty$; we henceforth will $\boxed{\text{only consider } \mathfrak{D} \geq 0.}$ Courtesy the substitutions $2K + 2 := \mathfrak{D}$ and $t := r^2$, we have that

$$I_\mathfrak{D} = \frac{1}{2}\int_0^\infty \underbrace{\mathrm{e}^{-r^2}}_{\mathrm{e}^{-t}}\, \underbrace{r^{2K}}_{t^K}\, \underbrace{2r\,\mathrm{d}r}_{\mathrm{d}t} = \frac{1}{2}\int_0^\infty \mathrm{e}^{-t}\, t^K\,\mathrm{d}t.$$

This RhS suggests defining a fnc which is traditionally called the ***Gamma function***:

$*$: $$\Gamma(K+1) := \int_0^\infty t^K \mathrm{e}^{-t}\,\mathrm{d}t, \quad \textit{for } K \in \mathbb{C} \textit{ with } \mathrm{Re}(K) > -1.$$

So $I_{2K+2} = \frac{1}{2} \cdot \Gamma(K+1)$. Hence $I_\mathfrak{D} = \frac{1}{2} \cdot \Gamma(\frac{\mathfrak{D}}{2})$. Solving for $\sigma_{\mathfrak{D}-1}$ in (2.1b) thus yields

1b: $$\sigma_{\mathfrak{D}-1} = \frac{\pi^{\mathfrak{D}/2}}{I_\mathfrak{D}} = 2 \cdot \frac{\pi^{\mathfrak{D}/2}}{\Gamma(\mathfrak{D}/2)}.$$

Computing $\boldsymbol{\Gamma}()$. Easily, $\boldsymbol{\Gamma}(1) = \int_0^\infty \mathrm{e}^{-t}\,\mathrm{d}t = 1$. When $\mathrm{Re}(K)>0$, then $\lim_{t\searrow 0}[t^K\mathrm{e}^{-t}]$ is zero. Thus

$$t^K[-\mathrm{e}^{-t}]\Big|_{t=0}^{t=\infty} = 0\,.$$

Integrating RhS(∗) by parts produces $\boldsymbol{\Gamma}(K+1) = 0 - \int_0^\infty[-\mathrm{e}^{-t}]\cdot Kt^{K-1}\,\mathrm{d}t$. Thus

For all $K\in\mathbb{C}$ *with* $\mathrm{Re}(K)>0$: $\boldsymbol{\Gamma}(K+1) = K\cdot\boldsymbol{\Gamma}(K)\,.$

This and (1b) yield $\sigma_{\mathfrak{D}+1} = \sigma_{\mathfrak{D}-1}\cdot\frac{2\pi}{\mathfrak{D}}$. Combined with (1), we now have

1c:
$$\sigma_{\mathfrak{D}} = \sigma_{\mathfrak{D}-2}\cdot\frac{2}{\mathfrak{D}-1}\cdot\boldsymbol{\pi}\,,$$
$$\beta_{\mathfrak{D}} = \beta_{\mathfrak{D}-2}\cdot\frac{2}{\mathfrak{D}}\cdot\boldsymbol{\pi}\,.$$

for each $\mathfrak{D}\in[2\,..\,\infty)$.

SurArea and Vol of the $\mathfrak{D}$-ball

We know that $\sigma_0 = 2$ [an interval has 2 endpts] and that $\sigma_1 = 2\boldsymbol{\pi}$. So (1c) allows us to fill-in the $\sigma_{\mathfrak{D}}$ column below. Then (1) allows us to fill-in the $\beta_{\mathfrak{D}}$ column.

$\mathfrak{D}$	$\beta_{\mathfrak{D}}\cdot r^{\mathfrak{D}}$	$\sigma_{\mathfrak{D}}\cdot r^{\mathfrak{D}}$
0	1	2
1	$2\cdot r$	$2\boldsymbol{\pi}\cdot r$
2	$\boldsymbol{\pi}\cdot r^2$	$4\boldsymbol{\pi}\cdot r^2$
3	$\frac{4}{3}\boldsymbol{\pi}\cdot r^3$	$2\boldsymbol{\pi}^2\cdot r^3$
4	$\frac{1}{2}\boldsymbol{\pi}^2\cdot r^4$	$\frac{8}{3}\boldsymbol{\pi}^2\cdot r^4$
5	$\frac{8}{15}\boldsymbol{\pi}^2\cdot r^5$	$\boldsymbol{\pi}^3\cdot r^5$
6	$\frac{1}{6}\boldsymbol{\pi}^3\cdot r^6$	$\frac{16}{15}\boldsymbol{\pi}^3\cdot r^6$
7	$\frac{16}{105}\boldsymbol{\pi}^3\cdot r^7$	$\frac{1}{3}\boldsymbol{\pi}^4\cdot r^7$
8	$\frac{1}{24}\boldsymbol{\pi}^4\cdot r^8$	$\frac{32}{105}\boldsymbol{\pi}^4\cdot r^8$
9	$\frac{32}{945}\boldsymbol{\pi}^4\cdot r^9$	$\frac{1}{12}\boldsymbol{\pi}^5\cdot r^9$
10	$\frac{1}{120}\boldsymbol{\pi}^5\cdot r^{10}$	$\frac{64}{945}\boldsymbol{\pi}^5\cdot r^{10}$
11	$\frac{64}{10395}\boldsymbol{\pi}^5\cdot r^{11}$	$\frac{1}{60}\boldsymbol{\pi}^6\cdot r^{11}$
12	$\frac{1}{720}\boldsymbol{\pi}^6\cdot r^{12}$	$\frac{128}{10395}\boldsymbol{\pi}^6\cdot r^{12}$

We can get a non-recursive formula for $\sigma_{\mathfrak{D}}$ using factorial notation. [Exercise.]

$\boldsymbol{\Gamma}()$ summary. *Function*

2.2b: $\boldsymbol{\Gamma}(z) := \int_0^\infty t^{z-1}\mathrm{e}^{-t}\,\mathrm{d}t\,,$ *for* $z\in\mathbb{C}$ *with* $\mathrm{Re}(z)>0$,

has values

2.2c:
$$\boldsymbol{\Gamma}(0) = \infty\,;$$
$$\boldsymbol{\Gamma}(\tfrac{1}{2}) = \sqrt{\boldsymbol{\pi}}\,;$$
$$\boldsymbol{\Gamma}(1) = 1\,;$$
$$\boldsymbol{\Gamma}(n) = [n-1]!\,,\quad \textit{for } n\in\mathbb{Z}_+\textit{. More generally,}$$
$$\boldsymbol{\Gamma}(z) = [z-1]\cdot\boldsymbol{\Gamma}(z-1)\,,\quad \textit{for } \mathrm{Re}(z)>1.$$

Centroids

Our goal here is to find the centroid of a hemi-ball and hemi-sphere. Measuring from the center of the radius-r ball/sphere of dimension $\mathfrak{D}$, define constants $\mathbf{B}_{\mathfrak{D}}$ and $\mathbf{S}_{\mathfrak{D}}$ by

$\mathbf{B}_{\mathfrak{D}}\cdot r$:= Dist-to-centroid of the $\mathfrak{D}$-hemi-ball;

$\mathbf{S}_{\mathfrak{D}}\cdot r$:= Dist-to-centroid of the $\mathfrak{D}$-hemi-sphere.

We will compute this using an integral in polar coords of an angle θ. For $K\neq -1$, note that

3:
$$\int_0^{\pi/2}\sin(\theta)^K\cos(\theta)\cdot\mathrm{d}\theta = \frac{1}{K+1}\,.$$

Henceforth use $\mathbf{c}$ and $\mathbf{s}$ for $\cos(\theta)$ and $\sin(\theta)$. For the $[\mathfrak{D}+1]$-hemi-ball

$$\mathrm{Vol} = \int_0^{\pi/2}\underbrace{\beta_{\mathfrak{D}}\cdot[r\mathbf{s}]^{\mathfrak{D}}}_{\text{Area of slice.}}\cdot\underbrace{\mathbf{s}\cdot r\,\mathrm{d}\theta}_{\text{Thickness of slice.}}\,.$$

We already know that the value of this integral is $\frac{1}{2}\beta_{\mathfrak{D}+1}\cdot r^{\mathfrak{D}+1}$. The *Total Torque Relative to Zero* is computed by the same integral, together with a lever arm.

$$TTRZ = \int_0^{\pi/2}\underbrace{\beta_{\mathfrak{D}}\cdot[r\mathbf{s}]^{\mathfrak{D}}}_{\text{Area.}}\cdot\underbrace{\mathbf{s}\cdot r\,\mathrm{d}\theta}_{\text{Thickness.}}\cdot\underbrace{r\mathbf{c}}_{\text{Lever arm.}}\,.$$

Courtesy (3),

4: $$TTRZ \;=\; r^{\mathfrak{D}+2}\beta_{\mathfrak{D}} \cdot \frac{1}{\mathfrak{D}+2}\,.$$

The quotient gives the centroid.

5: $$\mathbf{B}_{\mathfrak{D}+1} \;=\; \frac{\text{TTRZ}}{\text{Vol}} \;=\; \frac{\beta_{\mathfrak{D}}}{\beta_{\mathfrak{D}+1}} \cdot \frac{2}{\mathfrak{D}+2}\,.$$

♦

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