Staple!

Ord: ___ ___

Plex
MAA4402 8436

Class-Prac-V

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Notation. All sets are subsets of $\mathbb{C}$.
For sets B and E, the difference set is

$$B \smallsetminus E \quad := \quad \left\{ x \in B \;\middle|\; x \notin E \right\}.$$

The complement of E is $E^c := \mathbb{C} \smallsetminus E$.
For short-answer: Write **DNE** if the object does not exist or the operation cannot be performed. NB: **DNE** $\neq \{\} \neq 0$.

Prac-V1: Short answer. Show no work. $\mathbb{C}$-plane

a Blanks$\in\mathbb{R}$. So $\frac{1}{2+3\mathbf{i}} =$ ______ $+\mathbf{i}\cdot$ [______].

Thus $\dfrac{5-\mathbf{i}}{2+3\mathbf{i}} =$ ______ $+\mathbf{i}\cdot$ [______].

Soln: Note $\omega := 2+3\mathbf{i}$ is not zero. Thus $\frac{1}{\omega} = \frac{\overline{\omega}}{\omega\overline{\omega}} = \frac{\overline{\omega}}{|\omega|^2}$. So $\dfrac{1}{\omega} = \dfrac{\mathrm{Re}(\omega)}{|\omega|^2} + \mathbf{i}\cdot\dfrac{-\mathrm{Im}(\omega)}{|\omega|^2} = \dfrac{2}{13} + \mathbf{i}\cdot\dfrac{-3}{13}$.

With $\alpha := 5 - \mathbf{i}$, note $|\omega|^2 \cdot \dfrac{\alpha}{\omega} = \overline{\omega}\alpha = 7 - 17\mathbf{i}$. Whence

$$\frac{\alpha}{\omega} \;=\; \frac{7}{13} + \mathbf{i}\frac{-17}{13}.$$

For z complex, $\mathrm{Im}(z) = \mathrm{Formula}(z, \overline{z}) = \dfrac{z - \overline{z}}{2\mathbf{i}}$.

b Complex number $[x + \mathbf{i}y]^2 = -9\mathbf{i}$, for *real numbers* $x>y$, where $x = +\dfrac{3}{\sqrt{2}}$ and $y = -\dfrac{3}{\sqrt{2}}$.

Sqroot: Since $-\mathbf{i} = \exp(\mathbf{i}\cdot\frac{-\pi}{2})$, one sqroot of $-\mathbf{i}$ is $\exp(\mathbf{i}\cdot\frac{-\pi}{4}) \overset{\text{note}}{=\!=} \frac{1-\mathbf{i}}{\sqrt{2}}$. Multiplying by $\sqrt{9} = 3$ gives a particular sqroot of $-9\mathbf{i}$, namely $\frac{3}{\sqrt{2}} \cdot [1 - \mathbf{i}]$.

c *Reals* $x=$ ______ and $y=$ ______ where $x + \mathbf{i}y = [1+\mathbf{i}]^{166}$. [*Hint: Multiplying complexes multiplies their moduli, and adds their angles.*]

Power-soln. As $[1+\mathbf{i}]^2 = 2\mathbf{i}$, our $\alpha := [1+\mathbf{i}]^{166}$ equals $[2\mathbf{i}]^{83}$. Note $\mathbf{i}^{83} = \mathbf{i}^3 = [-\mathbf{i}]$. Thus $\alpha = 2^{83} \cdot [-\mathbf{i}]$.

d Distance $\left|e^{\mathbf{i}[\pi/4]} - 2\mathbf{i}\right| =$ ______.

Soln: With $R := \frac{1}{\sqrt{2}}$ [Reciprocal], note $e^{\mathbf{i}\cdot\frac{\pi}{4}} = [1+\mathbf{i}]R$. With $\mathbf{D} := \left|[1+\mathbf{i}]R - 2\mathbf{i}\right|$, then $\mathbf{D}^2$ equals

$$[R-0]^2 + [R-2]^2 = R^2 + [R^2 + 4 - 4R]$$
$$= 1 + 4 - 4R = 5 - 2\sqrt{2}.$$

Thus, $\mathbf{D} = \sqrt{5 - 2\sqrt{2}}$.

e Value $\left|e^{\mathbf{i}\theta} - \mathbf{i}\right| = 2$, where angle $\theta =$ ______.

Soln: Points $\mathbf{i}$ and $e^{\mathbf{i}\theta}$ are on the unit-circle, C. [This, since θ is an angle, thus is real]. The diameter of C is 2. The only C-point at distance 2 from $\mathbf{i}$ is its antipodal point, $-\mathbf{i}$. Thus $\theta = -\frac{\pi}{2}$.

f The ***Laplacian*** of a twice-differentiable fnc $h{:}\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, is $\frac{d}{dx}\left(\frac{dh}{dx}\right) + \frac{d}{dy}\left(\frac{dh}{dy}\right) \overset{\text{notation}}{=\!=} h_{xx} + h_{yy}$.

g Fnc $u(x,y) := \cos(y\cdot x) - 7x$ maps $\mathbb{R}\times\mathbb{R}\to\mathbb{R}$. Its Laplacian is $[\Delta(u)](x,y) = -\cos(y \cdot x) \cdot [y^2 + x^2]$.

There exists function $v{:}\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ such that $f(x+\mathbf{i}y) := u(x,y) + \mathbf{i}v(x,y)$ is holomorphic. T (F)

Lapl Soln: Let $\mathcal{C} := \cos(y\cdot x)$ and $\mathcal{S} := \sin(y \cdot x)$.
Note $\Delta(u) = \Delta(\mathcal{C}) = \mathcal{C}_{xx} + \mathcal{C}_{yy}$, since the $7x$ is killed-off by 2nd-differentiation.
The Chain rule gives $\mathcal{C}_x = -\mathcal{S}\cdot y$. Hence $\mathcal{C}_{xx} = -\mathcal{C}\cdot y^2$. As $\mathcal{C}$ is symmetric in x and y, we get for free that $\mathcal{C}_{yy} = -\mathcal{C}\cdot x^2$. Adding, $\Delta(u) = \Delta(\mathcal{C}) = -\cos(y\cdot x)\cdot[y^2 + x^2]$.
As $\Delta(u) \not\equiv \mathbf{0}$, our u is not harmonic. Hence, by our thm from class, this u is *not* the real part of a holomorphic fnc.

h The std.pic. of $\mathbb{C}$ is called the ________ plane.

Set $w := \log(3)\cdot[2+\mathbf{i}]$. So $|w|=$ ________.

$\text{Re}(e^w)=$ ________ and $|e^w|=$ ________.

i Write $\cos(-2\mathbf{i})$, which is real, ITOf exp() and *finite* add/sub/mul/div: $\cos(-2\mathbf{i})=$ ________.

And $\cos(-2\mathbf{i})$ lies in [circle the correct interval]

$(-\infty, \frac{-1}{5}]$ $(\frac{-1}{5}, \frac{1}{5}]$ $(\frac{1}{5}, 2]$ $(2,5]$ $(5,15]$ $(15,45]$ $(45,\infty)$

Soln: Recall that $\cos(z) = [e^{iz} + e^{-iz}]/2$, for all $z \in \mathbb{C}$. Hence $\cos(-2\mathbf{i})$ equals

$$\frac{1}{2}e^2 + \frac{1}{2}e^{-2}.$$

The $e^{-2}/2$ term is negligible, here. As $2 < e < 3$, so $4 < e^2 < 9$. Thus $2 < \frac{1}{2}e^2 < \frac{9}{2}$.

j Value $\text{Log}([\mathbf{i}e]^3)=$ ________.

Soln: In polar form, $[\mathbf{i}e]^3 = -e^3\mathbf{i}$ is $e^3\cdot\text{cis}(-\frac{\pi}{2})$. So its Log is $3 - \frac{\pi}{2}\mathbf{i}$, since $\log(e^3) = 3$.

[P.V of $[1+\mathbf{i}]^{\mathbf{i}}$] $= r\cdot\exp(\mathbf{i}\theta)$, where $r=$ ________

and $\theta=$ ________, with $r>0$ and θ real.

P.V: As $1+\mathbf{i} = \sqrt{2}\cdot\text{cis}(\frac{\pi}{4})$, its Log is $\log(\sqrt{2}) + \frac{\pi}{4}\mathbf{i}$. So

$$\mathbf{i}\cdot\text{Log}(1+\mathbf{i}) = -\frac{\pi}{4} + \mathbf{i}\cdot\log(\sqrt{2}).$$

Applying exp() says $[1+\mathbf{i}]^{\mathbf{i}}$ equals $r\cdot\exp(\mathbf{i}\theta)$, where

$$r = e^{-\pi/4} \quad \textit{and} \quad \theta = \log(\sqrt{2}) = \log(2)/2.$$

Prac-V2: Short answer. Metric space stuff

k On a set Ω, a ***metric*** is a map $d: \Omega \to [0,\infty)$ such that $\forall w,x,y,z \in \Omega$:

MS1: $d(w,x) = 0$ IFF $w = x$; [Zero-distance exactly when ***same point***.].

MS2: $d(z,y) = d(y,z)$; [Metric d is ***symmetric***.].

MS3: $d(w,y) \le d(w,x) + d(x,y)$; [Metric d satsifies the ***triangle inequality***.].

l A subset $S \subset \mathbb{C}$ is ***path-connected*** if ________.

... if <u>for *all*</u> points $p,q \in S$, <u>*there exists*</u> a map $f: [0,1] \to S$ which is **continuous**, with $f(0) = p$ and $f(1) = q$.

m *The empty-set is connected:* (T) F

Punctured ball $\text{PBal}_2(3\mathbf{i})$ *is connected:* (T) F

$\text{Sph}_2(5\mathbf{i}) \cap \text{Sph}_2(\mathbf{i})$ *is connected:* (T) F

$\text{Sph}_2(4\mathbf{i}) \cup \text{Sph}_2(-\mathbf{i})$ *is connected:* T (F)

$\text{Sph}_2(5\mathbf{i}) \cup \text{CldBal}_2(\mathbf{i})$ *is closed:* (T) F

Soln: In $\mathbb{C}$, a punctured-ball is path-connected, so is certainly connected.

Since $|5\mathbf{i} - \mathbf{i}| = 2\cdot 2$, intersection $\text{Sph}_2(5\mathbf{i}) \cap \text{Sph}_2(\mathbf{i})$ is singleton $\{3\mathbf{i}\}$; connected. But $|4\mathbf{i} - -\mathbf{i}| > 2\cdot 2$, so *union* $\text{Sph}_2(4\mathbf{i}) \cup \text{Sph}_2(-\mathbf{i})$ comprises two *not*-touching circles, hence is disconnected.

Recall that the union of *finitely many* closed sets is closed. Hence $\text{Sph}_2(5\mathbf{i}) \cup \text{CldBal}_2(\mathbf{i})$ is closed.

n All these sets are non-empty: Sets U and V are open. Sets K, E and E_n are closed. Sets S and T are each connected.

Set $U \smallsetminus K$ is open: **AT** AF Nei

Set $U \cup K$ is open: AT AF **Nei**

Set $E \cap K$ is closed: **AT** AF Nei

Union $\bigcup_{n=1}^{\infty} E_n$ is closed: AT AF **Nei**

$\exists q \in [S \cap T]$; so $S \cup T$ is connected: **AT** AF Nei

$S \cup T$ *is* connected. [Were "*connected*" replaced by the stronger "*path-connected*", this problem would be almost automatic.]

With $\mathbf{B} := S \cup T$, consider $\mathbb{C}$-open sets $\mathcal{U} \sqcup \mathcal{V} \supset \mathbf{B}$. [Recall that $\sqcup$ is our *disjoint-union* symbols; so $\mathcal{U} \cap \mathcal{V}$ is empty.]

Since open sets $\mathcal{U},\mathcal{V}$ have $\mathcal{U} \sqcup \mathcal{V} \supset S$, the connectedness of S forces either $\mathcal{U} \sqcap S$ (disjointness) or $\mathcal{V} \sqcap S$.

WLOG, $\mathcal{V} \ni q$; so $\mathcal{V} \cap S$ is non-void, whence $\mathcal{U} \sqcap S$. The same argument applies to connected set T, since $T \ni q$. Thus $\mathcal{U} \sqcap T$. Hence pair $(\mathcal{U},\mathcal{V})$ does <u>not</u> separate $\mathbf{B}$, as $\mathcal{U}$ misses $\mathbf{B}$ altogether.

o Let $S := \mathrm{PBal}_2(3\mathbf{i})$.

Its boundary is $\partial(S) =$ $\mathrm{Sph}_2(3\mathbf{i}) \cup \{3\mathbf{i}\}$.

[You may use our ball/sphere notation as well as $\cup$, $\cap$, complement and set-braces, to describe your answer.]

Prac-V3: Short answer. Binomials/multinomials.

p Binomial coefficient $\binom{7}{4} = \binom{7}{3} = \frac{7\cdot 6\cdot 5}{3\cdot 2\cdot 1} = 35$.

q Multinomial coefficient $\binom{9}{4,2,3} =$ ______ $=$ ______.

[*Note:* Write your ans. ITOf factorials, then **also** write it as a single integer, or product of two, **without** factorials.]

Nomial Soln: Directly, $\binom{9}{4,2,3} = \frac{9!}{4!\cdot 2!\cdot 3!}$. Computing,
$$\binom{9}{4,2,3} = \binom{9}{4}\cdot\binom{5}{2} = \frac{9\cdot 8\cdot 7\cdot 6}{4\cdot 3\cdot 2\cdot 1}\cdot\frac{5\cdot 4}{2\cdot 1}.$$
Hence, $\binom{9}{4,2,3} = 9\cdot 2\cdot 7\cdot 5\cdot 2 = [63\cdot 2]\cdot 10 = 1260.$

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(multinom-coeff 9 '(4 2 3))  =>  1260
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r Compute the real $\alpha =$ ______ such that
$$*: \quad 3^{\alpha}\cdot\sum_{k=0}^{4000}\binom{4000}{k}2^k = \sum_{j=0}^{1995}\binom{1995}{j}8^j.$$
[*Hint:* The Binomial Theorem]

Binom Soln: LhS($*$) equals $3^{\alpha}\cdot[2+1]^{4000} = 3^{\alpha+4000}$. RhS($*$) equals $[8+1]^{1995} \stackrel{\text{note}}{=} 3^{[2\cdot 1995]}$. Consequently,
$$\alpha = [2\cdot 1995] - 4000 = -10.$$

s As a single numeral, ______ is the following alternating sum:
$$*: \ 1 - 3\cdot\binom{9}{1} + 9\cdot\binom{9}{2} - 27\cdot\binom{9}{3} + 81\cdot\binom{9}{4} - \ldots - 3^9\cdot\binom{9}{9}.$$
[*Hint:* First determine: Is the value positive, zero, or negative.]

Soln: Courtesy the Binomial theorem, this sum equals $[1+{-3}]^9$, i.e $[-2]^9 = -512$.

Details: Sum ($*$) equals $\sum_{k=0}^{9} 1^{9-k}\cdot[-3]^k\cdot\binom{9}{k}$.

Prac-V4: Short answer. LFT = Linear-fractional transformation.

t With $f(z) := \frac{3z+2}{2z+5}$, then $f^{-1}(z) := \frac{az+b}{cz+d}$ where

a= ________ , b= ________ , c= ________ , d= ________ .

u Cross-ratio $[z,\ 2+\mathbf{i},\ 4\mathbf{i},\ 5] = \frac{az+b}{cz+d}$ where

a= ________ , b= ________ , c= ________ , d= ________ .

v The point $p := 2+7\mathbf{i}$ goes, under stereographic projection, to (x, y, z) on the RS, where

$x=$ ____________ , $y=$ ____________ , $z=$ ____________ .

OYOP: *In grammatical English **sentences**, write your essays on every 2nd line* (usually)*, so I can easily write between the lines.*

Essay-V5: Below, $h:\mathbb{C}\to\mathbb{C}$.

α Suppose h is differentiable at the point $3+2\mathbf{i}$. Writing h in real and imaginary parts, $h(x+\mathbf{i}y) = u(x,y) + \mathbf{i}v(x,y)$, *state* the Cauchy-Riemann eqns for h at $3+2\mathbf{i}$.

State C-R. Using u_x for $\frac{\partial u}{\partial x}$ etc., the C-R eqns for h at $3+2\mathbf{i}$ are $u_x(3,2) = v_y(3,2),\ u_y(3,2) = -v_x(3,2)$.

β Suppose h is differentiable at a point $z \in \mathbb{C}$. Carefully *derive* the Cauchy-Riemann eqns, directly from the defn of "differentiable".

Derive C-R. Firstly, for h to be diff'able at z means: *Our h is defined in a nhbd of z, and* $\lim_{\Delta z\to 0} \frac{h(z+\Delta z)-h(z)}{\Delta z}$ *exists* in $\mathbb{C}$.

Let $w := h(z)$ and $\Delta w := h(z+\Delta z) - h(z)$.

CASE: Pure real: $\Delta z := \Delta x$ Computing, Δw equals

$$u(x+\Delta x,\ y) + \mathbf{i}v(x+\Delta x,\ y) - \big[u(x,y) + \mathbf{i}v(x,y)\big]$$
$$= \big[u(x+\Delta x,\ y) - u(x,y)\big] + \mathbf{i}\big[v(x+\Delta x,\ y) - v(x,y)\big]\,.$$

Hence, $\frac{\Delta w}{\Delta z}$ equals

$$\frac{u(x+\Delta x,\ y) - u(x,y)}{\Delta x} + \mathbf{i}\cdot\frac{v(x+\Delta x,\ y) - v(x,y)}{\Delta x}\,.$$

Sending $\Delta x \to 0$ yields that

$$\dagger\!: \quad \lim_{\Delta z\to 0}\frac{\Delta w}{\Delta z} = u_x(x,y) + \mathbf{i}\cdot v_x(x,y)\,.$$

CASE: Pure imag: $\Delta z := \mathbf{i}\Delta y$ Our Δw equals

$$\big[u(x,\ y+\Delta y) - u(x,y)\big] + \mathbf{i}\big[v(x,\ y+\Delta y) - v(x,y)\big]\,.$$

So $\frac{\Delta w}{\Delta z}$ equals $\frac{u(x,y+\Delta y)-u(x,y)}{\mathbf{i}\Delta y} + \mathbf{i}\cdot\frac{v(x,y+\Delta y)-v(x,y)}{\mathbf{i}\Delta y}$, i.e

$$-\mathbf{i}\cdot\frac{u(x,\ y+\Delta y) - u(x,y)}{\Delta y} + \frac{v(x,\ y+\Delta y) - v(x,y)}{\Delta y}\,.$$

Launching $\Delta y \to 0$ reveals that

$$\ddagger\!: \quad \lim_{\Delta z\to 0}\frac{\Delta w}{\Delta z} = -\mathbf{i}\cdot u_y(x,y) + v_y(x,y)\,.$$

The C-R equations are $\mathrm{Re}(\mathrm{RhS}(\dagger)) = \mathrm{Re}(\mathrm{RhS}(\ddagger))$ and $\mathrm{Im}(\mathrm{RhS}(\dagger)) = \mathrm{Im}(\mathrm{RhS}(\ddagger))$, equating Re & Im parts. QED

Essay-V6: For <u>*reals*</u> α, P, Q, ω, consider equation

†: $$\alpha[x^2 + y^2] + Px + Qy + \omega \;=\; 0$$

in $\mathbb{R}\times\mathbb{R}$. Show that (†) describes a ***gen-circle*** [i.e, a *circle*-or-*line*; a ***generalized-circle***] IFF

∗: $$P^2 + Q^2 \;>\; 4\alpha\omega\,.$$

V1:	___ ___ ___	000pts
V2:	___ ___ ___	000pts
V3:	___ ___ ___	000pts
V4:	___ ___ ___	000pts
V5:	___ ___ ___	000pts
V6:	___ ___ ___	000pts
Total:	___	0pts

HONOR CODE: *"I have neither requested nor received help on this exam other than from my professor (or his colleague)."*

Ord:

. . . .