Soln to 4×4 TicTacToe
team essay take-home problem

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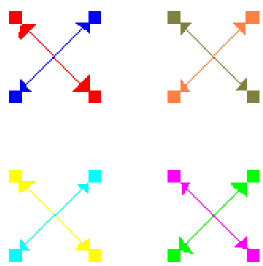
The question was:

Q: The 4×4 TTT (TicTacToe) board is $\mathbb{B} := [1 \,..\, 4] \times [1 \,..\, 4]$; sixteen ***cells***. Let Γ denote the TTT-automorphism group; the set of self-bijections of $\mathbb{B}$ which preserve all ten TTTs. So $\mathtt{R},\mathtt{F} \in \Gamma$, where $\mathtt{R}$ rotates $\mathbb{B}$ by 90° CCW, and $\mathtt{F}$ flips $\mathbb{B}$ about its vertical axis. Evidently, $\langle \mathtt{R}, \mathtt{F} \rangle_\Gamma \cong \mathbb{D}_4$.

A less obvious TTT-automorphism is the ***swizzle***, $\mathtt{S}$: It exchanges each corner square with the central square that it (diagonally) touches; and it does The Right Thing on the edge squares.



i Easily, $\mathtt{S} \leftrightarrows \mathtt{R}$ and $\mathtt{S} \leftrightarrows \mathtt{F}$, hence each element of subgroup $\Lambda := \langle \mathtt{S}, \mathtt{F}, \mathtt{R} \rangle$ can be written in form $\mathtt{S}^b\,\mathtt{R}^c\,\mathtt{F}^d$ for integers b, c, d. So $|\Lambda| =$ ______________; prove this.

Draw a "16-dot picture" (4×4 dots, with arrows), for each element $\alpha \in \Lambda$. However, use the same picture for α rotated or flipped about any line, or for α^{-1} (rotated or flipped). *Label* each picture with *all* the automorphisms that it describes. The total number of labels should equal your $|\Lambda|$.

ii Find a TTT-aut $\mathtt{T}$ which is **not** in the Λ subgp. Write the commutation relations between $\mathtt{T}$ and each of $\{\mathtt{S}, \mathtt{R}, \mathtt{F}\}$. Prove that each aut α can be written as $\alpha = \mathtt{T}^{a_1}\mathtt{S}^{a_2}\mathtt{R}^{a_3}\mathtt{F}^{a_4}$, with $a_i \in \mathbb{Z}$. For element $\beta = \mathtt{T}^{b_1}\mathtt{S}^{b_2}\mathtt{R}^{b_3}\mathtt{F}^{b_4}$, give an explicit multiplication rule showing how to compute the exponents $\{c_i\}_{i=1}^4$ of $\beta\alpha = \mathtt{T}^{c_1}\mathtt{S}^{c_2}\mathtt{R}^{c_3}\mathtt{F}^{c_4}$.

Prove that $\langle \mathtt{T}, \mathtt{S}, \mathtt{R}, \mathtt{F} \rangle$ is **all** of Γ. Thus $|\Gamma| =$ _______.

Draw all the new labeled 16-dot pictures for $\Gamma \smallsetminus \Lambda$.

iii Find a set of *involutions* which generates Γ. Compute (with proof) the center of Γ; what is its order?

iv With $\mathbf{u} \in \mathbb{B}$ the upper-LH corner of $\mathbb{B}$, define its stabilizer $\Upsilon := \mathrm{Stab}_\Gamma(\mathbf{u})$. With proof, compute $|\Upsilon| =$ _______. The number of Υ-orbits is __________.

Soln

We have commutation relation $\mathtt{FR} = \mathtt{R}^3\mathtt{F}$. Swizzle $\mathtt{S}$ commutes with $\mathtt{F}$ and $\mathtt{R}$, so the general elt in Λ has form $\mathtt{S}^b\mathtt{R}^c\mathtt{F}^d$, where $b,d \in \{0,1\}$ and $c \in \{0,1,2,3\}$. Thus $|\Lambda| = 2{\cdot}4{\cdot}2 = 16$.

The ***weight*** of a cell, Wei($\mathbf{c}$), is the number of TTTs owning $\mathbf{c}$. In our 4×4 TTT, Wei(Corner) = 3, Wei(Edge) = 2, Wei(Center) = 3. So there might be an auto carrying corners to centers.

The ***rank*** of a TTT is the number of varying coordinates. On our 4×4, the rank-1 TTTs are vertical and horizontal. The two diagonal-TTTs each have rank-2. Can an auto change rank?

Rank is preserved. The ***weight-multiset*** [multiset of its cell-weights] is $\{3,3,2,2\}$ for a vert/horiz TTT, but is $\{4,4,4,4\}$ for a diagonal TTT.

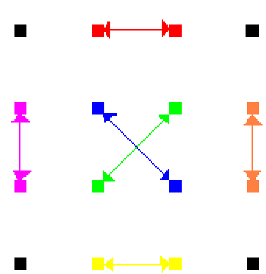
***All* autos.** Fix a mystery auto β, acting on board

$\mathbf{u}$	p	q	v
P	c_1	c_2	?
Q	d_1	d_2	?
V	?	?	z

It must carry upper-LH-cell $\mathbf{u}$ to some weight-3 cell; and we can use $\mathtt{S},\mathtt{R},\mathtt{F}$ to carry it back: *WLOG β fixes* $\mathbf{u}$. Hence β sends the $\mathbf{u}$—v TTT to either itself (possibly scrambling the cells), or sends it to $\mathbf{u}$—V. But $\mathbb{D}_4$ owns the flip on the $\mathbf{u}$—z diagonal; *so WLOG β sends $\mathbf{u}$—v to itself.* Thus β sends $\mathbf{u}$—V to itself.

As β preserves weight, necessarily β fixes v and V; consequently, β fixes z.

Either β fixes p and q, or it exchanges them. If those edge-cells are fixed, then the frozen 4 corners force $\beta = \mathit{Id}$. Conversely, if p and q are exchanged, then the frozen u and z forces c_1 to be exchanged with d_2. Continuing, β is seen to be…



…The ***Traffic light***, named by one of my former students. Define $\mathtt{T}$ to be it.

Composing automorphisms, we find:

$*$: *Each of* $\mathtt{T},\mathtt{S}$ *commutes with each of* $\mathtt{R},\mathtt{F}$. *Also,* $\mathtt{ST} = \mathtt{TSR}^2$, *and* $\mathtt{FR} = \mathtt{R}^3\mathtt{F}$. *Each of* $\mathtt{T},\mathtt{S},\mathtt{F}$ *has order* 2, *and* $\mathtt{R}$ *has order* 4.

Hence every elt of subgroup $\langle \mathtt{T},\mathtt{S},\mathtt{F},\mathtt{R}\rangle$ can be written as $\mathtt{T}^a\mathtt{S}^b\mathtt{R}^c\mathtt{F}^d$, where $a,b,d \in \{0,1\}$ and $c \in \{0,1,2,3\}$; The foregoing β-argument shows that subgroup $\langle \mathtt{T},\mathtt{S},\mathtt{F},\mathtt{R}\rangle$ is *all* of Γ; whence $|\Gamma| = 2{\cdot}2{\cdot}4{\cdot}2 = 32$.

Involutions. This set $\{\mathtt{T},\ \mathtt{S},\ \mathtt{RF},\ \mathtt{F}\}$ of involutions generates Γ, since $\langle \mathtt{RF},\ \mathtt{F}\rangle = \langle \mathtt{R},\mathtt{F}\rangle$.

Center of Γ. Each Γ-elt can be written as sg, where $s \in \{\varepsilon, \mathtt{T}, \mathtt{S}, \mathtt{TS}\}$, and $g \in \langle \mathtt{R},\mathtt{F}\rangle$.

Suppose $sg \in \mathcal{Z}(\Gamma)$. Since g commutes with $\mathtt{T}$ and $\mathtt{S}$, so must s. Our $(*)$ now forces $s = \varepsilon$.

Thus g must commute with both $\mathtt{R}$ and $\mathtt{F}$. Hence $\mathcal{Z}(\Gamma) = \{\varepsilon,\ \mathtt{R}^2\}$.

Pictures

Let's count auto-pictures, up to flipping or rotating the image, or reversing the arrows.

Pic is of	$\mathbb{D}_4$ conjugates	Inverses	Count
Id	*None*	*Self-inverse*	1
$\mathtt{R}^2$	*None*	*Self-inverse*	1
$\mathtt{R}$	$\mathcal{J}_{\mathtt{F}}(\mathtt{R}) = \mathtt{R}^3$	*Each other*	2
$\mathtt{F}$	$\mathcal{J}_{\mathtt{R}}(\mathtt{F}) = \mathtt{R}^2\,\mathtt{F}$	*Self-inverse*	2
$\mathtt{RF}$	$\mathcal{J}_{\mathtt{R}}(\mathtt{RF}) = \mathcal{J}_{\mathtt{F}}(\mathtt{RF}) = \mathtt{R}^3\mathtt{F}$	*Self-inverse*	2

To get *all* elts, we can multiply this table on the left by each element $s \in \{\mathtt{S}, \mathtt{T}, \mathtt{TS}\}$. Given elts $\mathbf{c},\mathbf{d} \in \langle \mathtt{R}, \mathtt{F}\rangle$, note that conjugating by $\mathbf{c}$ produces

$$\mathcal{J}_{\mathbf{c}}(s\mathbf{d}) \overset{\text{note}}{=\!=} s \cdot \mathcal{J}_{\mathbf{c}}(\mathbf{d})\,, \quad \textit{since } s \leftrightarrows \mathbf{c}.$$

So for the left two columns of the table, we can just multiply by s. As for inverses, note

$$[s\mathbf{d}]^{-1} \;=\; \mathbf{d}^{-1}s^{-1} \overset{\text{note}}{=\!=} s^{-1}\mathbf{d}^{-1}\,.$$

When s is an involution, then, the latter is $s\mathbf{d}^{-1}$; we can just multiply the third column as well.

The upshot is that for $s = \mathtt{S}$ and $s = \mathtt{T}$, we can just multiply on the left. For $s = \mathtt{TS}$, we'll need to pay attention to inverses, since $\mathtt{TS}$ is not an involution.

Multiplying by S.

We obtain

Pic is of	$\mathbb{D}_4$ conjugates	Inverses	Count
$\mathtt{S}$	*None*	*Self-inverse*	1
$\mathtt{SR}^2$	*None*	*Self-inverse*	1
$\mathtt{SR}$	$\mathtt{SR}^3$	*Each other*	2
$\mathtt{SF}$	$\mathtt{SR}^2\,\mathtt{F}$	*Self-inverse*	2
$\mathtt{SRF}$	$\mathtt{SR}^3\mathtt{F}$	*Self-inverse*	2

The ten pictures of this and the preceding table, show the 16 autos comprising group Λ.

Multiplying by T.

For free, we have

Pic is of	$\mathbb{D}_4$ conjugates	Inverses	Count
$\mathtt{T}$	*None*	*Self-inverse*	1
$\mathtt{TR}^2$	*None*	*Self-inverse*	1
$\mathtt{TR}$	$\mathtt{TR}^3$	*Each other*	2
$\mathtt{TF}$	$\mathtt{TR}^2\,\mathtt{F}$	*Self-inverse*	2
$\mathtt{TRF}$	$\mathtt{TR}^3\mathtt{F}$	*Self-inverse*	2

The next case is slightly different, as $\mathtt{TS}$ is not an involution.

Multiplying by TS.

Note that $s := \mathtt{TS}$ has order 4, since $s^4 = [\mathtt{R}^2]^2 = \varepsilon$. So for $\mathbf{d}$ an element of $\mathbb{D}_4$,

$$[s\mathbf{d}]^{-1} \;=\; \mathbf{d}^{-1}s^{-1} \overset{\text{note}}{=\!=} s^3\mathbf{d}^{-1} \overset{\text{note}}{=\!=} \mathtt{TSR}^2\mathbf{d}^{-1}\,.$$

In this case, the 2nd row gets combined with the 1st:

Pic is of	$\mathbb{D}_4$ conjugates	Inverses	Count
$\mathtt{TS}$	*None*	$[\mathtt{TS}]^{-1} = \mathtt{TSR}^2$	2
$\mathtt{TSR}$	$\mathtt{TSR}^3$	*Self-inverse*	2
$\mathtt{TSF}$	$\mathtt{TSR}^2\,\mathtt{F}$	*Each other*	2
$\mathtt{TSRF}$	$\mathtt{TSR}^3\mathtt{F}$	*Each other*	2

Stabilizer subgroup. Here are few ways to see that $\Upsilon := \mathrm{Stab}_{\Gamma}(\mathbf{u})$ has order 4.

Already Λ carries a corner to every corner/center; and edge-cells have the wrong weight. So the Γ-orbit of a corner/center has size 8, whence, courtesy of our Orb-Stab thm, the stabilizer Υ has cardinality $\frac{|\Gamma|}{8} = 4$.

Secondly, our *All* autos proof gave us two binary degrees of freedom, once cell $\mathbf{u}$ was fixed. The two edge cells could either be fixed, or exchanged. And then, we could either flip the picture on the $\mathbf{u}$—z diagonal or not. So we have $2 \cdot 2 = 4$ possibilities. Indeed, this shows that Υ *is a copy of Klein-4*.

We could just write down those 4 auts, or we can scan the pictures looking for where upper-left is fixed. Either way gives that

$$\Upsilon = \{\varepsilon, \mathtt{T}, \mathtt{RF}, \mathtt{TRF}\} = \langle \mathtt{T}, \mathtt{RF} \rangle .$$

The orbit-structure of Υ:
Two fixed-points; Three 2-cycles; Two 4-cycles.
Hence there are $2 + 3 + 2 = 7$ many Υ-orbits.
[*Consistency check*: $[2 \cdot 1] + [3 \cdot 2] + [2 \cdot 4] = 16$.]