# Taylor's theorem in several guises

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***Preliminaries.*** A polynomial $p()$ is an "***N*-topped polynomial**" if $\mathrm{Deg}(p) < N$. So $x^2 - 2x$ and $x + \sqrt{7}$ are 3-topped, but $x^3 + x - 5$ is not. The set of $N$-topped polynomials is an $N$-dimensional VS.[♡1]

## Taylor polynomials

The setting is an open interval $J \subset \mathbb{R}$. Use $\mathsf{Diff}^N = \mathsf{Diff}^N(J \to \mathbb{R})$ for the set of $N$-times differentiable fncs. This is a superset of

$$\mathbf{C}^N \quad = \quad \mathbf{C}^N(J \to \mathbb{R})\,;$$

those, whose $N^{\text{th}}$ derivative is *continuous*.

For posint $N$, a point $Q \in J$ and $f \in \mathsf{Diff}^{N-1}$, define

1a: *"the $N^{th}$* ***Taylor polynomial*** *of $f$, centered at $Q$"*

to be the *unique* $N$-topped polynomial $p()$ whose *zero*$^{\text{th}}$ through $[N-1]^{st}$ derivatives, at $Q$, agree with those of $f$. I.e, these $N$ equations hold:

$$p(Q) = f(Q), \quad p'(Q) = f'(Q), \quad p''(Q) = f''(Q), \quad \ldots, \quad p^{(N-1)}(Q) = f^{(N-1)}(Q)\,.$$

One easily checks that $p(x)$ must be the RhS of (1b), below. That is,

1b: $$\mathbf{T}_N(x) \;=\; \mathbf{T}^f_{N,Q}(x) \;:=\; \sum_{k\in[0\,..\,N)} \frac{f^{(k)}(Q)}{k!}\cdot[x-Q]^k$$

is the $N^{\text{th}}$ Taylor polynomial of $f$.

Define the "the $N^{th}$ ***remainder term*** of $f$", written $\mathbf{R}^f_{N,Q}$ or just $\mathbf{R}_N$, by

1c: $$f(x) \;=:\; \mathbf{T}_N(x) + \mathbf{R}_N(x)\,.$$

[♡1] Abbreviations: VS for *vector space*; FTC for *Fundamental Theorem of Calculus*; posint for *positive integer*.

**Properties of $\mathbf{T}_N$.** Use $\mathbf{T}_{N,Q}[f]$ as a synonym for $\mathbf{T}^f_{N,Q}$, with the same convention for the $\mathbf{R}_N$ operator and friends. Often times either the fnc, $f$, or the center-point, $Q$, is implicit, and we drop it from the notation.

For a scalar $\alpha$ and $f,g \in \mathsf{Diff}^{N-1}$, note that

2: $$\mathbf{T}_N[\alpha\cdot f] \;=\; \alpha\cdot\mathbf{T}_N[f] \quad\text{and}\quad \mathbf{T}_N[f+g] \;=\; \mathbf{T}_N[f] + \mathbf{T}_N[g]\,.$$

I.e, $\mathbf{T}_N$ is a *linear operator* from $\mathsf{Diff}^{N-1}$ to the VS of polynomials.

**3: Lemma.** *For $f \in \mathsf{Diff}^N$, and $Q,x \in J$:*

3a: $$\mathbf{T}^{f'}_N \;=\; \Big[\mathbf{T}^f_{N+1}\Big]'. \quad \textit{Also,}$$

3b: $$\int_Q^x \mathbf{T}^{f'}_{N,Q}(t)\cdot \mathrm{d}t \;=\; \mathbf{T}^f_{N+1,Q}(x) \;-\; f(Q)\,, \quad \textit{and}$$

3c: $$\int_Q^x \mathbf{R}^{f'}_{N,Q}(t)\cdot \mathrm{d}t \;=\; \mathbf{R}^f_{N+1,Q}(x)\,.$$ ♢

*Note.* Our (3a) says $\mathbf{T}_N$ ***almost*** commutes with differentiation, $\mathbf{D}$ [The **Taylor-series** operator does commute with differentiation.]

Letting $\mathbf{I}$ denote the identity operator, we have that $\mathbf{I} = \mathbf{T}_N + \mathbf{R}_N$. Since $\mathbf{I}$ is linear and *almost* commutes with $\mathbf{D}$, so is/does $\mathbf{R}_N$. □

*Pf of (3a).* WLOG $Q{=}0$. By defn, $\mathbf{T}^{f'}_N(x)$ equals

$$\sum_{k=0}^{N-1} \frac{[f']^{(k)}(0)}{k!}\cdot x^k \;=\; \sum_{k=0}^{N-1} \frac{f^{(k+1)}(0)}{[k+1]!}\cdot\frac{\mathrm{d}}{\mathrm{d}x}[x^{k+1}]$$

$$\underset{\text{setting } \ell := k+1}{=} \frac{\mathrm{d}}{\mathrm{d}x}\Bigg[\sum_{\ell=1}^{N} \frac{f^{(\ell)}(0)}{\ell!}\cdot x^\ell\Bigg]\,.$$

This indeed equals $\frac{\mathrm{d}}{\mathrm{d}x}[\mathbf{T}^f_{N+1}(x)]$, since the $\frac{\mathrm{d}}{\mathrm{d}x}$ kills-off the constant $\ell{=}0$ term. ♦

*Pf of (3b).* Courtesy FTC, integrating (3a) gives

$$\int_Q^x \mathbf{T}^{f'}_{N,Q}(t)\cdot \mathrm{d}t \;=\; \mathbf{T}^f_{N+1,Q}(x) - \mathbf{T}^f_{N+1,Q}(Q) \;=\; \mathbf{T}^f_{N+1,Q}(x) \;-\; f(Q)\,.$$ ♦

## Taylor's Theorem for $\mathbb{R}$

We produce two estimates[♡2] of the remainder term. In the results below, $N \in \mathbb{Z}_+$.

**4: Lem.** *Fix an $f \in \mathsf{Diff}^N(J{\to}\mathbb{R})$. At each $x \in J$, function $q \mapsto \mathbf{T}^f_{N,q}(x)$ is differentiable, with value*

4a: $$\frac{\mathrm{d}}{\mathrm{d}q}\mathbf{T}^f_{N,q}(x) \;=\; f^{(N)}(q)\cdot\frac{[x-q]^{N-1}}{[N-1]!}.$$ *Thus,*

4b: $$\frac{\mathrm{d}}{\mathrm{d}q}\mathbf{R}^f_{N,q}(x) \;=\; -\frac{\mathrm{d}}{\mathrm{d}q}\mathbf{T}^f_{N,q}(x)\,.$$ ♢

*Proof.* WLOG $x{=}7$. For natnum $k$ and posint $\ell$, define

$$A_k(q) \;:=\; \frac{f^{(k)}(q)}{k!}\cdot[7-q]^k, \quad\text{and}$$
$$B_\ell(q) \;:=\; \frac{f^{(\ell)}(q)}{[\ell-1]!}\cdot[7-q]^{\ell-1}.$$

Now $A_0(q) = f(q){\cdot}1$, so $\frac{\mathrm{d}}{\mathrm{d}q}A_0(q) = f'(q) = B_1(q)$. And for $k$ positive, $\frac{\mathrm{d}}{\mathrm{d}q}A_k(q)$ equals

$$\frac{1}{k!}\Big[f^{(k+1)}(q){\cdot}[7-q]^k + f^{(k)}(q){\cdot}k{\cdot}[7-q]^{k-1}{\cdot}[-1]\Big]$$
$$\stackrel{\text{note}}{=\!=}\; B_{k+1}(q) \;-\; B_k(q)\,.$$

[The above $[-1]$ is $\frac{\mathrm{d}}{\mathrm{d}q}[7-q]$.] Thus $\frac{\mathrm{d}}{\mathrm{d}q}$ of $\mathbf{T}^f_{N,q}(7)$ is (notationally dropping the "$q$")

$$[B_N - B_{N-1}] + [B_{N-1} - B_{N-2}] + \cdots + [B_2 - B_1] + B_1\,.$$

Telescoping, this equals $B_N$, which is RhS$(4a)$. [Exer.: Prove (4b) from (4a), in two sentences.] ♦

**5: TayThm-1.** *Consider a fnc $f \in \mathsf{Diff}^N(J{\to}\mathbb{R})$. For distinct points $Q$,$x \in J$, there exists a point $\mathbf{c}{=}\mathbf{c}_x{=}\mathbf{c}_{N,Q,x,f}$ strictly between $Q$ and $x$, such that*

5a: $$\mathbf{R}^f_{N,Q}(x) \;=\; f^{(N)}(\mathbf{c}_x)\cdot\frac{[x-Q]^N}{N!}.$$

*I.e,* $$f(x) = \Big[\sum_{k=0}^{N-1} f^{(k)}(Q){\cdot}\frac{[x-Q]^k}{k!}\Big] + f^{(N)}(\mathbf{c}_x){\cdot}\frac{[x-Q]^N}{N!}.$$ ♢

*Pf.* WLOG $Q=3$ and $x=7$. Define $M \in \mathbb{R}$ by

5b: $$\mathbf{R}_{N,3}(7) \;=\; M\cdot\frac{[7-3]^N}{N!}.$$

(Possible, since $[7-3]^N \neq 0$.) Create a function

5c: $$\varphi(q) \;:=\; \mathbf{R}_{N,q}(7) - M\cdot\frac{[7-q]^N}{N!}.$$

Our "multiplier" $M$ was defined so that $\varphi(3)=0$. And $\varphi(7) = 0 - M{\cdot}0 = 0$. (We used that $[7-7]^N{=}0^N$ is zero, since $N$ is positive.) [Exer.: Why is $\mathbf{R}_{N,7}(7)=0$?]

Rolle's thm now applies to $\varphi$ on $[3,7]$, asserting a point $\mathbf{c} \in (3,7)$ st. $\varphi'(\mathbf{c})=0$. By Lemma 4,

$$0 \;=\; -f^{(N)}(\mathbf{c})\cdot\frac{[7-\mathbf{c}]^{N-1}}{[N-1]!} \;-\; M{\cdot}\frac{[7-\mathbf{c}]^{N-1}}{[N-1]!}\cdot[-1] \quad.$$ ♦

Since $\mathbf{c} \neq 7$, quantity $[7-\mathbf{c}]^{N-1}$ is not zero; so we may divide, to conclude that $0 = -f^{(N)}(\mathbf{c}) + M$. From (5b), then, $\mathbf{R}_{N,3}(7) \;=\; f^{(N)}(\mathbf{c})\cdot\frac{[7-3]^N}{N!}$

**6: TayThm-2.** *For $f \in \mathbf{C}^N(J)$ and points $Q$,$x \in J$:*

6a: $$\mathbf{R}^f_{N,Q}(x) \;=\; \frac{1}{[N-1]!}\int_Q^x f^{(N)}(s)\cdot[x-s]^{N-1}\,\mathrm{d}s\,.$$ ♢

*Pf.* WLOG $Q=3$ and $x=7$, making our goal

†: $$\mathbf{R}_{N,3}(7) \;\stackrel{?}{=}\; \frac{1}{[N-1]!}\int_3^7 f^{(N)}(s)\cdot[7-s]^{N-1}\,\mathrm{d}s\,.$$

Function $\theta(s) := \mathbf{T}_{N,s}(7)$ is $\mathbf{C}^1$, since $f \in \mathbf{C}^N$. Note $\theta(7) = f(7)$, so

$$\mathbf{R}_{N,3}(7) \;\stackrel{\text{def}}{=\!=}\; f(7) - \mathbf{T}_{N,3}(7)$$
‡: $$= \theta(7) - \theta(3) \;=\; \int_3^7 \theta'(s)\,\mathrm{d}s$$

by FTC, since $\theta'$ is continuous on $J$. And RhS(‡) equals RhS(†), courtesy Lemma 4. ♦

[♡2] The TayThm-1 estimate, (5a), is called *Lagrange's form* of the $N^{\text{th}}$ remainder term.

Our TayThm-2, (6a), is one of several *Integral forms* of the remainder.

*Aside: An incorrect formula.* The 1st edition of Creighton Buck's text has (6a), but the the 3rd ed. (P.148, top) asserts that $\mathbf{R}^f_{N,Q}(x)$ equals

$$\text{6b:}\quad W^f_{N,Q}(x) := \frac{1}{[N-1]!}\int_Q^x f^{(N)}(s)\cdot[s-Q]^{N-1}\,\mathrm{d}s\,.$$

Just because RhS(6b) differs from RhS(6a) doesn't mean that it is wrong; there are several formulae for $\mathbf{R}_N$ ITOf integrals. The following example, however, indeed shows that $W^f_{N,Q} \neq \mathbf{R}^f_{N,Q}$.

Let $f := [z \mapsto z^3]$, $Q := 0$ and $N := 2$. Thus $\mathbf{T}_2(x)$ equals $0 + 0\cdot x$, so $\boxed{\mathbf{R}_2(x) = x^3}$. Since $f''(s) = 3\cdot 2\cdot s$, our $W_2(x)$ equals

$$*:\quad \frac{1}{1!}\int_0^x 3\cdot 2\cdot s\cdot[s-0]\cdot\mathrm{d}s \;=\; 2s^3\Big|_{s=0}^{s=x} \;=\; 2x^3\,.$$

So $W_2(1) = 2 \neq 1 = \mathbf{R}_2(1)$. In contrast, RhS(6a) equals $\int_0^x 3\cdot 2\cdot s\cdot[x-s]\cdot\mathrm{d}s$, which equals

$$\Big[x\cdot\int_0^x 3\cdot 2\cdot s\cdot\mathrm{d}s\Big] - \text{RhS}(*) \;=\; x\cdot 3x^2 - 2x^3\,.$$

And this, happily, equals $x^3 \overset{\text{note}}{=\!=} \mathbf{R}_2(x)$. □

## Applications of Taylor's theorem

Here are five uses.

**7: *Translating a polynomial.*** Suppose you wish to express polynomial

$$p(x) \;=\; C_0 + C_1 x + \cdots + C_{N-1}x^{N-1}$$

in form $\sum_{k\in[0..N)} B_k\cdot[x-6]^k$. You *could* solve a system of $N$ many linear eqns in $N$ unknowns.

But $p$ **equals** its $N^{\text{th}}$ TayPoly (centered anywhere we want). So each $B_k$ is the $k^{\text{th}}$ coeff of $\mathbf{T}^p_{N,6}$. Thus $B_k$ equals $p^{(k)}(6)/k!$. □

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**8: Limits.** *Suppose $f \in \mathsf{Diff}^2(\mathbb{R}\to\mathbb{R})$ and $\lim_{x\to\infty} f(x)$ exists in $\mathbb{R}$, and $\limsup_{x\to\infty}\big|f''(x)\big| < \infty$. Then $f'(x) \to 0$ as $x\nearrow\infty$.* ◇

*Set-up for both proofs.* WLOG $\boxed{*:\ \lim_{x\to\infty} f(x) = 0}$ and $|f''()| \leq 8$. □

*Pf.* Fixing $\varepsilon>0$, we will produce an $L\in\mathbb{R}$ for which:

$$\forall x \geq L:\quad |f'(x)| \leq 5\varepsilon\,.$$

Consider TayPoly $\mathbf{T}^f_{2,x}(y) = f(x) + f'(x)[y-x]$ for $N=2$. The TayThm-1 says for each pair $x<y$ there is a point $\mathbf{c}_{x,y} \in (x,y)$ for which

$$f(y) \;=\; f(x) + f'(x)[y-x] \;+\; \frac{f''(\mathbf{c}_{x,y})}{2}[y-x]^2.$$

Solve for $f'(x)$. Since $|f''(\mathbf{c}_{x,y})| \leq 8$,

$$|f'(x)| \;\leq\; \left|\frac{f(y)-f(x)}{y-x}\right| + \frac{8}{2}\cdot[y-x]\,.$$

By $(*)$, there is $L \in \mathbb{R}$ st. $|f| \leq \varepsilon^2/2$ on $[L,\infty)$. Setting $y = x+\varepsilon$, we have that for each $x \geq L$:

$$|f'(x)| \;\leq\; \left|\frac{\varepsilon^2}{y-x}\right| + 4\cdot[y-x]$$
$$\leq\; \frac{\varepsilon^2}{\varepsilon} + 4\varepsilon \;=\; 5\varepsilon\,.$$ ♦

*Proof, barehands.* Could $\beta := \liminf_{x\to\infty} f'(x) > 0$? No; for then the graph of $f$ would grow at some minimum rate, thus could not have a horizontal asymptote. Hence $\beta \leq 0$. Similarly $\alpha \geq 0$, where $\alpha := \limsup_{x\to\infty} f'(x)$. The upshot: $\alpha \geq 0 \geq \beta$.

FTSOC, suppose $\alpha > \beta$. WLOG $\alpha > 0$; otherwise, replace $f$ by $-f$. Set $\varepsilon := \alpha/3$.

Let $a_1$ be, say, the smallest non-negative "$x$-value" such that $f(a_1) \geq 2\varepsilon$. Since

$$\alpha \;>\; 2\varepsilon \;>\; \varepsilon \;>\; \beta\,,$$

the following process never stops: Take $b_n$ to be the smallest "$x$" exceeding $a_n$ for which $f(b_n) \leq \varepsilon$. Take $a_{n+1}$ to be the smallest value exceeding $b_n$ for which $f(a_{n+1}) \geq 2\varepsilon$.

Let $J_n := [a_n, b_n]$. Each restriction $f'\lfloor_{J_n} \geq \varepsilon$, for $n = 1, 2, \ldots$. By FTC,

$$\int_{b_n}^{a_n} f'' \;=\; f'(a_n) - f'(b_n) \;\geq\; 2\varepsilon - \varepsilon \;=\; \varepsilon\,.$$

So $\varepsilon \leq \int_{J_n}|f''| \leq 8\cdot[b_n - a_n]$. Hence each $b_n - a_n$ dominates $\frac{8}{\varepsilon}$; thus the numbers $a_n, b_n \nearrow \infty$.

By $(*)$, then, $f(b_n) - f(a_n) \to 0$, as $n\nearrow\infty$. But

$$f(b_n) - f(a_n) \;=\; \int_{a_n}^{b_n} f' \;\geq\; \int_{a_n}^{b_n}\varepsilon$$
$$\geq\; [b_n - a_n]\varepsilon \;\geq\; \frac{8}{\varepsilon}\cdot\varepsilon \;=\; 8.$$ ♦

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*$N^{th}$-derivative test for extrema.* With $J \subset \mathbb{R}$ an open interval and $f \in \mathbf{C}^\infty(J \to \mathbb{R})$, suppose $Q \in J$ is a ***critical point*** for $f$, ie. $f'(Q) = 0$. We seek a test determining if $f$ has a *strict local-max/min* at $Q$. To this end, define $\text{MinNZD}(f, Q)$ to be

9a: $$\inf\left\{k \in \mathbb{Z}_+ \;\middle|\; f^{(k)}(Q) \neq 0\right\} \overset{\text{note}}{\in} [1 .. \infty]. \quad \square$$

**9b: Min/Max Prop'n.** With $J, f, Q$ as above: *Suppose $N := \text{MinNZD}(f, Q)$ is in $[2 .. \infty)$. When $N$ is...*

*...even: If $f^{(N)}(Q) > 0$ then $f$ has a strict-local-min; else, $f$ has a strict-local-max, at $Q$.*

*...odd: Then $f$ has a strict-SignChange at $Q$.* ◊

*Pf.* Let $V := f^{(N)}(Q)$, which is not zero. Since $f^{(N)}$ is continuous at $Q$, there is a small open interval $I \ni Q$ for which $f^{(N)}\downharpoonright_I$ has the same sign as $V$. Our Prop'n will be established by (9c), below.

Let $\mathbf{T} := \mathbf{T}^f_{N,Q}$ and $\mathbf{R} := \mathbf{R}^f_{N,Q}$. Recall that $f^{(k)}(Q) = 0$, for each $k \in [1 .. N)$. For each $x$, then, $\mathbf{T}(x) = f(Q)$, and thus $\mathbf{R}(x) = f(x) - f(Q)$.

For $x \neq Q$, there is a point $\tau_x$, between $x$ and $Q$, with $\mathbf{R}(x) = \frac{1}{N!} f^{(N)}(\tau_x) \cdot [x - Q]^N$. Taking $x \in I$ forces $\tau_x \in I$, so

$f^{(N)}(\tau_x)$ *has the same sign as* $V$.

But $f(x) - f(Q) = \mathbf{R}(x)$, so the sign-fnc gives

9c: $$\text{Sgn}\big(f(x) - f(Q)\big) = \text{Sgn}(V) \cdot [\text{Sgn}(x - Q)]^N,$$ for each $x \in I$ with $x \neq Q$. ♦

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**10: Lemma.** *Set $J := [-1, 1]$. Imagine a function $f \in \text{Diff}^3(J \to \mathbb{R})$ with*

10a: $$f(0) = 0 = f'(0).$$

10b: $$f(-1) = 0 \quad \text{and} \quad f(1) = 1.$$

*Then there exists $\tau \in J^\circ$ with $f'''(\tau) \geq 3$.* ◊

*Counting degrees-of-freedom.* What would it mean if each derivative $\{f^{(k)}\}_{k=3}^\infty$ were identically-zero? Then $f$ is 3-topped polynomial (i.e, at most quadratic) and so comes from a 3-dim'al VS.

But (10a,10b) are *four* conditions, making it plausible that we cannot fulfill them when constrained to the VS of 3-topped polynomials. □

*Proof.* ISTProduce points $\alpha, \beta \in J^\circ$ with

10c: $$f^{(3)}(\alpha) + f^{(3)}(\beta) \geq 6.$$

Let $\mathbf{T} := \mathbf{T}^f_{3,0}$ and $\mathbf{R} := \mathbf{R}^f_{3,0}$. Define a number $M$ by

$$\mathbf{T}(x) = f(0) + f'(0)x + \overbrace{\frac{f''(0)}{2}}^{M} x^2 \overset{\text{by (10a)}}{=\!=} M \cdot x^2.$$

Thus $\mathbf{T}(\pm 1) = M \cdot [\pm 1]^2 = M$. From (10b), then,

$$1 = f(1) - f(-1) = [M + \mathbf{R}(1)] - [M + \mathbf{R}(-1)] = \mathbf{R}(1) - \mathbf{R}(-1).$$

By TayThm-1, $\exists \alpha \in (-1, 0)$ and $\exists \beta \in (0, 1)$ with

$$\mathbf{R}(1) = \tfrac{1}{6} \cdot f^{(3)}(\beta) \cdot [+1 - 0]^3 = \tfrac{1}{6} \cdot f^{(3)}(\beta);$$
$$\mathbf{R}(-1) = \tfrac{1}{6} \cdot f^{(3)}(\alpha) \cdot [-1 - 0]^3 = -\tfrac{1}{6} \cdot f^{(3)}(\alpha).$$

This, using the previous display, yields (10c). ♦

**DiffyQ.** A fnc $f \in \mathsf{Diff}^2(\mathbb{R}\to\mathbb{R})$ with $f(7) = 0$ and $f'(7) = 0$, satisfies

$$\dagger\!: \qquad f'' + f \;=\; \mathbf{0}\,. \qquad \text{[I.e, the zero-fnc.]}$$

Prove that $f = \mathbf{0}$, using Taylor's thm, as follows:

First, show that $f \in \mathbf{C}^\infty$. Then, for a *fixed* $x_0 \in \mathbb{R}$, argue that $\left|f(x_0)\right|$ is as small as desired, by upper-bounding with Taylor-remainder terms from **Our Taylor's pamphlet** on the Teaching Page.

*Proof.* Let $\|\cdot\|$ mean $\|\cdot\|_{\sup}$. Induction on $n \in \mathbb{N}$ shows that each $f^{(n)}$ is diff'able. Moreover

$$\ddagger\!: \quad \forall k,\ell \in \mathbb{N}\colon \;\; \text{If } k \equiv_4 \ell \text{ then } f^{(k)} = f^{(\ell)}.$$

As $f^{(2)}(7) = -f(7) = 0$ and $f^{(3)}(7) = -f'(7) = 0$, property (‡) tells us that each $\mathbf{T}^f_{n,7}$ is the zero-function. In particular, with $\mathbf{R}_n$ denoting $\mathbf{R}^f_{n,7}$,

$$\forall n \in \mathbb{Z}_+\colon \quad \mathbf{R}_n(x_0) \;=\; f(x_0)\,.$$

***Upper bounds.*** Let $J$ be the compact interval going from 7 to $x_0$. Let $M_n := \|f^{(n)}\lfloor_J\|$. Define

$$\mathbf{M} \quad := \quad \text{Max}(M_0, M_1, M_2, M_3)\,.$$

Then $\mathbf{M}$ dominates each $\|f^{(n)}\lfloor_J\|$, courtesy (‡). From TayThm-2, then, $|f(x_0)|$ equals $|\mathbf{R}_n(x_0)|$ which equals

$$\begin{aligned}
&\tfrac{1}{[n-1]!}\cdot\left|\int_7^{x_0} f^{(n)}(t)\cdot[x_0 - t]^{n-1}\cdot \mathrm{d}t\right| \\
&\le \tfrac{1}{[n-1]!}\int_7^{x_0} |f^{(n)}(t)|\cdot|x_0 - t|^{n-1}\cdot \mathrm{d}t \\
&\le \tfrac{\mathbf{M}}{[n-1]!}\int_7^{x_0} |x_0 - t|^{n-1}\cdot \mathrm{d}t\,.
\end{aligned}$$

Let $L := \text{Len}(J)$. Then

$$|f(x_0)| \;\le\; \tfrac{\mathbf{M}}{[n-1]!}\int_7^{x_0} L^{n-1}\cdot \mathrm{d}t \;=\; \mathbf{M}\cdot\tfrac{L^n}{[n-1]!}\,.$$

This last goes to zero, as $n \nearrow \infty$. ♦

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## Radius of Convergence

*Series notations.* Customs about how "series" is used in the context of "convergence of a series" are a bit strange. A "***series*** $\vec{\mathbf{e}}$" is a <u>*sequence*</u> $\vec{\mathbf{e}} = (e_k)_{k=0}^{\infty}$, but[♡3] where the word "series" hints to the reader our interest in its <u>*sum*</u> $\sum(\vec{\mathbf{e}})$. This sum is the limit –when it exists– of the corresponding "partial-sum sequence" $\vec{\mathbf{s}}$, where

$$s_N := \sum_{k\in[0\,..\,N)} e_k\,.$$

Use $\boxed{\vec{\mathbf{s}} = \mathrm{P}\Sigma(\vec{\mathbf{e}})}$ to indicate this partial-sum relation between sequences. Here, phrase "series $\vec{\mathbf{e}}$ is convergent" means that $\lim(\vec{\mathbf{s}})$ exists and is finite. So $\sum(\vec{\mathbf{e}}) := \lim(\vec{\mathbf{s}})$.

To clarify, the $n^{\text{th}}$ partial sum means the sum of the first $n$ terms, regardless of the initial index. For example, suppose $\vec{\mathbf{b}} = (b_\ell)_{\ell=5}^{\infty}$, and $\vec{\mathbf{e}} = \mathrm{P}\Sigma(\vec{\mathbf{b}})$. Then $e_3 = b_5+b_6+b_7$, and $e_0 = 0$.

*Example*: Let $\vec{\mathbf{b}} := (k^2)_{k=1}^{\infty}$ and $\vec{\mathbf{a}} := \mathrm{P}\Sigma(\vec{\mathbf{b}})$. Then $a_n = \frac{1}{6}\cdot[2n^3 + 3n^2 + n]$. □

**11: Root-test lemma.** *Given a series $\vec{\mathbf{e}} \subset \mathbb{C}$, define*

$$*:\quad \Lambda := \limsup_{n\to\infty} \sqrt[n]{|e_n|} \overset{\text{note}}{\in} [0, +\infty]\,.$$

*If $\Lambda < 1$ then $\vec{\mathbf{e}}$ is an absolutely-convergent series.*

*If $\Lambda > 1$ then $\vec{\mathbf{e}}$ is "magnificently divergent" Not only $|e_n| \not\to 0$, but indeed $\limsup_{n\to\infty} |e_n| = +\infty$.* ◊

*Proof.* Let $a_n := |e_n|$,

$\boxed{\text{Case: When } \Lambda < 1.}$ ISTShow that $\vec{\mathbf{a}}$ is a convergent series. Pick $\rho$ with $\Lambda < \rho < 1$. Take $K$ large enough that $\sup_{n\ge K} \sqrt[n]{a_n} \le \rho$. Hence $\sum_{n\ge K} a_n \le \sum_{n\ge K} \rho^n < \infty$. And $\sum_{n\in[1\,..\,K]} a_n < \infty$.

$\boxed{\text{Case: When } \Lambda > 1.}$ Pick $\rho$ with $1 < \rho < \Lambda$. By $(*)$, the set $J := \left\{n \;\middle|\; \sqrt[n]{a_n} > \rho\right\}$ is infinite. And each $n \in J$ has $a_n > \rho^n$. ♦

[♡3] The index will usually start at zero, but it doesn't have to. The sequence $\vec{\mathbf{e}}$ might be $(e_k)_{k=24}^{\infty}$, or $(e_k)_{k=-5}^{\infty}$.

A function $f:\mathbb{R}\to\mathbb{R}$ is ***eventually positive*** if $\boxed{\exists K \text{ s.t } \forall x \ge K\colon f(x) > 0}$. Thus a degree-$k$ poly,

$$f(x) := C_k x^k + \cdots + C_1 x + C_0\,,$$

is eventually positive IFF $f$ has positive leading-coeff, $C_k > 0$.

*Power-series notation.* A sequence $\vec{\mathbf{c}} \subset \mathbb{C}$ and point $Q \in \mathbb{C}$ determine a ***power series***

$$\text{12a:}\quad \mathrm{PS}_{\vec{\mathbf{c}},Q}(z) := \sum_{n=0}^{\infty} c_n \cdot [z - Q]^n\,. \quad □$$

From the notation we sometimes drop the the center of expansion, just writing $\mathrm{PS}_{\vec{\mathbf{c}}}$. This is especially true when the center of expansion is $0\in\mathbb{C}$.

Use "PS" to abbreviate the phrase "power series". Use McS to abbrev ***Maclaurin Series***; a PS centered at $Q$=0. E.g $\mathrm{McS}_{\vec{\mathbf{c}}}(z) = \sum_{n=0}^{\infty} [c_n \cdot z^n]$.

*Radius of Convergence.* The set of $z \in \mathbb{C}$ for which RhS(12$a$) converges is called the "***set-of-convergence***". We write it $\mathrm{SoC}(\vec{\mathbf{c}}, Q)$

It will turn out that the SoC comprises an open ball, possibly of radius 0 or ∞, together with some of the points on the boundary of this ball. This open ***ball of convergence*** is written $\mathrm{BoC}(\vec{\mathbf{c}}, Q)$. Its radius is the ***radius of convergence*** of RhS(12$a$), and is written $\mathrm{RoC}(\vec{\mathbf{c}})$.[♡4] So $\mathrm{RoC} := \mathrm{RoC}(\vec{\mathbf{c}})$ is always a value in $[0, +\infty]$, and $\mathrm{BoC}(\vec{\mathbf{c}}, Q) = \mathrm{Bal}_{\mathrm{RoC}}(Q)$. □

**12b: RoC Lemma** (Cauchy, 1821. Hadamard, 1888.) *Contemplate power series $\mathrm{PS}_{\vec{\mathbf{c}},Q}$, as in* (12a). *Let*

$$\Omega := \limsup_{n\to\infty} \sqrt[n]{|c_n|} \overset{\text{note}}{\in} [0, +\infty].$$

*Then $\mathrm{RoC}(\vec{\mathbf{c}}) = 1/\Omega$ where, here, we interpret $\frac{1}{0}$ as $+\infty$ and $\frac{1}{+\infty}$ as 0.* ◊

[♡4] The argument to RoC is a *sequence*. So we can write the RoC of PS $f(x) := \sum_{n=0}^{\infty} n^2 x^n$ as $\mathrm{RoC}(n \mapsto n^2)$, but <u>**not**</u> as $\mathrm{RoC}(n^2)$ nor as $\mathrm{RoC}(f)$.

*Proof sketch.* Set $a_n := |c_n|$. ISTConsider convergence at a non-negative $x \in \mathbb{R}$. Applying the Root-test,

$$\limsup_{n\to\infty} \sqrt[n]{\left|c_n x^n\right|} = \limsup_{n\to\infty} [x \cdot \sqrt[n]{a_n}\,] = x \cdot \limsup_{n\to\infty} \sqrt[n]{a_n} = x \cdot \Omega =: \Lambda\,.$$

So $\Lambda$ is less/greater than 1, as $x$ is less/greater than $\frac{1}{\Omega}$. ♦

**13:** *Three examples.* ASIDE: On a set $\mathbf{\Omega}$, each subset $B \subset \mathbf{\Omega}$ engenders $\mathbf{1}_B$, the "***indicator function*** of $B$". It is the fnc $\mathbf{\Omega} \to \{0,1\}$ sending points in $B$ to 1, and pts in its complement, $B^c := \mathbf{\Omega} \smallsetminus B$, to 0. [So $\mathbf{1}_B + \mathbf{1}_{B^c}$ is constant-1.] E.g, $\mathbf{1}_{\text{Primes}}(5)=1$ and $\mathbf{1}_{\text{Primes}}(9)=0$.

Let's apply the above (12b). Define

$$\mathbb{P} := \text{Primes};\ D := \text{Odds};\ S := \{1+n^2 \mid n \in \mathbb{N}\}\,.$$

Consider this power series:

13a: $$\sum_{n=0}^{\infty} 3^n \cdot \mathbf{1}_{\mathbb{P}}(n) \cdot x^n = 9x^2 + 27x^3 + 243x^5 + \dots.$$

Its RoC is $1/3$, since there are $\infty$ly many primes.

A funkier PS, centered at 8, is

13b: $$\sum_{k=0}^{\infty} \left[3^k \cdot \mathbf{1}_D(k) + 4^k \cdot \mathbf{1}_S(k)\right] \cdot [x-8]^k\,.$$

Since $\sqrt[n]{3^n + 4^n} \xrightarrow{n} 4$, and $|S| = \infty$, the RoC is $\frac{1}{4}$.

Even more interesting is this PS:

13c: $$\sum_{n=0}^{\infty} \left[5^n \cdot \mathbf{1}_{\mathbb{P}}(n) \cdot \mathbf{1}_S(n)\right] \cdot x^n\,.$$

As of March2017, its RoC is unknown. If there are $\infty$ly many primes[♡5] of form $1+n^2$ (conjectured, but unproven) then RoC $= \frac{1}{5}$; otherwise RoC $= \infty$, and the PS is a polynomial. □

[♡5] For the curious, see Wikipedia on Landau's problems.

**14: Lemma.** *For each $K \in \mathbb{R}$: $\lim_{x \nearrow \infty} \sqrt[x]{x^K} = 1$. Moreoever, for each rational function $h() := \frac{p()}{q()}$ which is eventually positive, $\lim_{n \nearrow \infty} \sqrt[n]{h(n)} = 1$.*

*Proof.* Use L'Hôpital's rule. Etc. ◊

**15: Same-RoC lemma.** *Consider a sequence $\vec{\mathbf{c}} = (c_0, c_1, \dots) \subset \mathbb{C}$, and let RoC := RoC($\vec{\mathbf{c}}$). For each natnum $K$, and for each rational function $g \neq$ Zip, these coefficient sequences*

*i:* $(0, \overset{K}{\dots}, 0, c_K, c_{K+1}, c_{K+2}, \dots)$

*ii:* $(c_K, c_{K+1}, c_{K+2}, \dots)$

*iii:* $\left(g(n) \cdot c_n\right)_{n=0}^{\infty}$

*give rise to power-series with RoC = RoC.* ◊

*Proof sketch.* Parts (i) and (ii) follow from (12b). Part (iii) follows from (14) and (12b). ♦

**16:** *Diff/Integrate a PS.* We differentiate and integrate, term-by-term, the $G := \text{PS}_{\vec{\mathbf{c}},0}$ power-series:

16a: $$F(x) = \sum_{j=1}^{\infty} b_j \cdot x^j\,, \text{ where } b_j := \tfrac{1}{j} \cdot c_{j-1}.$$
$$G(x) = \sum_{k=0}^{\infty} c_k \cdot x^k\,.$$
$$H(x) = \sum_{\ell=0}^{\infty} d_\ell \cdot x^\ell, \text{ where } d_\ell := [\ell+1] \cdot c_{\ell+1}.$$

Lemma (15) tells us that the three PSes have the same RoC.

Observe that $\text{PS}_{\vec{\mathbf{d}}}$ is the term-by-term derivative of $\text{PS}_{\vec{\mathbf{c}}}$. And $\text{PS}_{\vec{\mathbf{b}}}$ is the term-by-term integral of $\text{PS}_{\vec{\mathbf{c}}}$. Does the same relation hold between the *functions* that these PSes determine? □

**16b: Term-by-term PS Theorem.** *Given a sequence $\vec{\mathbf{c}} \subset \mathbb{R}$, define sequences/fncs $\vec{\mathbf{b}}, \vec{\mathbf{d}}, F, G, H$ by (16a) and let RoC := RoC($\vec{\mathbf{c}}$). Then*

†: $$\text{RoC}(\vec{\mathbf{b}}) = \text{RoC} = \text{RoC}(\vec{\mathbf{d}})\,.$$

*With $B :=$ BoC($\vec{\mathbf{c}}$), moreover,*

‡: $$\forall z \in B:\quad F(z) = \int_0^z G\,.$$

*And $G$ is in $\mathbf{C}^\infty(B \to \mathbb{R})$, with $G' = H$.* ◊

16c: Coro. *Suppose* PS $G(x) := \sum_{j=0}^{\infty} c_j \cdot [x-Q]^j$ *has positive* RoC. *Then this* PS *is the Taylor series of* $G$*, centered at* $Q$. ♢

*Pf of* (16b). We'll establish that $G'{=}H$; the integral result (‡) follows analogously. ISTo fix a posreal $\rho <$ RoC, let $U := \text{Bal}_\rho(0)$, and prove $G'{=}H$ when *restricted to* $U$. We will apply the DUC Thm (Derivative uniform-convergence) from *notes-AdvCalc.pdf* to these fncs (defined *only* on $U$)

$$f_n(x) \quad := \quad \sum\nolimits_{j\in[0\,..\,n]} c_j x^j\,.$$

By definition of coeff-sequence $\vec{\mathbf{d}}$ from (16a),

$$f_n'(x) \quad = \quad \sum\nolimits_{k\in[0\,..\,n)} d_k x^k\,.$$

In order to show that seq $\big(f_n'\big)_{n=1}^{\infty}$ is sup-norm Cauchy, pick a number $V$ with $\rho < V <$ RoC.

Now $\frac{1}{V} > \limsup_{n\to\infty} \sqrt[n]{|d_n|}$ since, by (15), $\text{RoC}(\vec{\mathbf{d}})$ equals RoC. Thus there is an index $K$ with

$$\forall n \geq K\colon \quad \sqrt[n]{|d_n|} < \tfrac{1}{V}\,.$$

We henceforth only consider indices $n$ dominating $K$. For each $k \geq n$, then,

16d: $$|d_k| \quad \leq \quad 1/V^k\,.$$

***Sup-norm.*** For $x \in U$ and indices $\ell{>}n$,

$$f_\ell'(x) - f_n'(x) \;=\; \sum\nolimits_{k\in[n\,..\,\ell)} d_k x^k\,.$$

From (16d), then,

$$\Big|f_\ell'(x) - f_n'(x)\Big| \;\leq\; \sum_{k=n}^{\infty} \frac{|x|^k}{V^k}\,.$$

Since $U$ owns $x$,

$$\Big|f_\ell'(x) - f_n'(x)\Big| \;\leq\; \sum_{k=n}^{\infty} \frac{\rho^k}{V^k} \;=\; \Big[\frac{\rho}{V}\Big]^n \cdot C\,,$$

where $C$ is the positive constant $1/[1-\frac{\rho}{V}]$.

Taking a supremum over all $x \in U$ yields

16e: $$\Big\|f_\ell' - f_n'\Big\| \;\leq\; \Big[\frac{\rho}{V}\Big]^n \cdot C\,,$$

for each pair $\ell > n \geq K$. Sending $n\nearrow\infty$ sends $\text{RhS}(16e) \to 0$.

The limit $\lim_n f_n(0)$ exists, equaling $c_0$. Now apply the DUC Thm. [Derivative uniform convergence.] ♦

**A power-series with a new center.** We show that a function defined by a PS is analytic in its entire ball-of-convergence.

17: *The setting.* We have a point $P \in \mathbb{C}$ and a sequence $\vec{\mathbf{a}} \subset \mathbb{C}$ such that $\alpha \in (0, +\infty]$, where $\alpha := \operatorname{RoC}(\vec{\mathbf{a}})$. This engenders a $\mathbf{C}^\infty$-fnc from $\operatorname{Bal}_\alpha(P) \to \mathbb{C}$, by

17a: $$\mathcal{F}(z) := \sum_{k=0}^{\infty} a_k \cdot [z - P]^k .$$

Fix a new center $Q \in \mathbb{C}$ with $|Q - P| < \alpha$. Thus

17b: *$\beta \in (0, +\infty]$, where $\beta := \alpha - |Q - P|$. Moreoever, $\operatorname{Bal}_\beta(Q) \subset \operatorname{Bal}_\alpha(P)$.* □

18: New-center theorem. *Take $P, Q, \alpha, \beta, \vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ from (17). For each natnum $k$, this summation is absolutely convergent:*

18a: $$b_k := \sum_{N=k}^{\infty} a_N \cdot \binom{N}{k} \cdot Q^{N-k} \in \mathbb{C} .$$

*Moreoever, $\operatorname{RoC}(\vec{\mathbf{b}}) \geq \beta > 0$. This value*

18b: $$\mathcal{G}(z) := \sum_{k=0}^{\infty} b_k \cdot [z - Q]^k ,$$

18c: *agrees with $\mathcal{F}(z)$, for each $z \in \operatorname{Bal}_\beta(Q)$.*

*Lastly, for each natnum $k$,*

18d: $$b_k = \frac{1}{k!} \cdot \mathcal{F}^{(k)}(Q) .$$

*In other words, RhS(18b) is the Taylor series for $\mathcal{F}$, centered at $Q$.* ♢

*Proof.* WLOG $P = 0$. Fix a point $Z \in \operatorname{Bal}_\beta(Q)$. Writing $Z = Q + [Z - Q]$, its $N^{\text{th}}$-power is

$$Z^N = \sum_{k=0}^{N} \binom{N}{k} \cdot Q^{N-k} \cdot [Z - Q]^k .$$

Thus, since $Z \in \operatorname{Bal}_\alpha(P)$,

$$\begin{aligned} f(Z) &= \sum_{N=0}^{\infty} a_N \cdot Z^N \\ &= \sum_{N=0}^{\infty} \sum_{k=0}^{N} \underbrace{a_N \cdot \binom{N}{k} \cdot Q^{N-k} \cdot [Z - Q]^k}_{h_{N,k}} . \end{aligned}$$

This is a sum, in a certain order, over the set $H := \left\{ (N, k) \in \mathbb{N} \times \mathbb{N} \,\middle|\, N \geq k \right\}$. We need this sum to be absolutely convergent. The sum $\sum_{N=0}^{\infty} \sum_{k=0}^{N} |h_{N,k}|$ equals

$$*: \quad \sum_{N=0}^{\infty} \sum_{k=0}^{N} |a_N| \cdot \binom{N}{k} \cdot |Q|^{N-k} \cdot |Z - Q|^k = \sum_{N=0}^{\infty} |a_N| \cdot Y^N ,$$

where $Y := |Q| + |Z - Q|$. From $Z \in \operatorname{Bal}_\alpha(0)$ and (17b), we conclude that $Y < \alpha$. From the proof of Root-test lemma (11, P.6), the righthand side of $(*)$ is finite.

Since $\mathbf{S} := \sum_{N=0}^{\infty} \sum_{k=0}^{N} |h_{N,k}|$ is finite, we can reverse the order of summation and conclude that

$$\begin{aligned} \mathbf{S} &= \sum_{k=0}^{\infty} \sum_{N=k}^{\infty} |h_{N,k}| \\ &= \sum_{k=0}^{\infty} \left[ \sum_{N=k}^{\infty} |a_N| \cdot \binom{N}{k} \cdot |Q|^{N-k} \right] \cdot |Z - Q|^k . \end{aligned}$$

We *could* have chosen our $Z \neq Q$, thus allowing division by $|Z - Q|^k$. Hence, each bracketed sum is finite. So each sum in (18a) is *absolutely convergent*, and we have a well-defined number $b_k$.

For a general $Z \in \operatorname{Bal}_\alpha(0)$, reversing the original sum gives

$$\begin{aligned} f(Z) &= \sum_{k=0}^{\infty} \sum_{N=k}^{\infty} h_{N,k} \\ &= \sum_{k=0}^{\infty} \left[ \sum_{N=k}^{\infty} a_N \cdot \binom{N}{k} \cdot Q^{N-k} \right] \cdot [Z - Q]^k , \end{aligned}$$

which equals $\sum_{k=0}^{\infty} b_k \cdot [Z - Q]^k$.

***Establishing*** **(18d).** Corollary 16c tells us that

$$k! \cdot b_k \stackrel{\text{by (16c)}}{=\!=} \mathcal{G}^{(k)}(Q) \stackrel{\text{by (18c)}}{=\!=} \mathcal{F}^{(k)}(Q) .$$

Hence (18d). ♦

19: Prop'n. *Power-series*

$$*: \qquad \mathcal{F}(z) \;:=\; \sum_{n=0}^{\infty} a_n\cdot[z-Q]^n$$

*has positive* RoC. *Suppose* $\vec{\mathbf{y}}$ *is a sequence of distinct complex numbers converging to* $Q$*, such that*

$$\forall j \in \mathbb{Z}_+ : \quad \mathcal{F}(y_j) \;=\; 0\,.$$

*Then* $\vec{\mathbf{a}}$ *is all-zero, and* $\mathcal{F}$ *is the zero function.* ♢

*Proof.* WLOG, each $y_j \neq Q$. FTSOC, suppose $\vec{\mathbf{a}} \neq \vec{\mathbf{0}}$; let $L$ be the smallest index with $a_L \neq 0$. Formally dividing $(*)$ by $[z-Q]^L$ gives PS

$$\mathcal{G}(z) \;:=\; \sum_{k=0}^{\infty} b_k\cdot[z-Q]^k\,,$$

where each $b_k := a_{L+k}$. Each $y_j - Q \neq 0$, so

$$\mathcal{G}(y_j) \;=\; \mathcal{F}(y_j)/[y_j - Q]^L \;=\; 0\,.$$

But $\mathsf{RoC}(\vec{\mathbf{b}}) = \mathsf{RoC}(\vec{\mathbf{a}}) > 0$, so $\mathcal{G}$ is cts in a nbhd of $Q$, and thus $\mathcal{G}(Q) = \lim\bigl(\mathcal{G}(\vec{\mathbf{y}})\bigr) = 0$. This contradicts that $\mathcal{G}(Q) = b_0 = a_L \neq 0$. ♦

20: PS Uniqueness Thm. *Imagine power-series*

$$\mathcal{F}(z) \;:=\; \sum_{n=0}^{\infty} a_n\cdot[z-P]^n \quad \text{and}$$
$$\mathcal{G}(z) \;:=\; \sum_{n=0}^{\infty} b_n\cdot[z-P]^n$$

*where* $B := \mathsf{BoC}(\vec{\mathbf{a}}) \cap \mathsf{BoC}(\vec{\mathbf{b}})$ *is non-void. Suppose there is a set* $Y \subset B$ *st.* $\mathcal{F}\lfloor_Y = \mathcal{G}\lfloor_Y$*, and* $Y$ *has a cluster point,* $Q_0$*, <u>in B</u>. Then* $\vec{\mathbf{a}} = \vec{\mathbf{b}}$*, so* $\mathcal{F} = \mathcal{G}$*.* ♢

*Remark.* It does <u>not</u> suffice for $Y$ to have a cluster-point on the *boundary* of $B$: Distinct functions $\mathcal{F}(z) := \sin(\frac{1}{z-7})$ and $\mathcal{G} := -\mathcal{F}$ have Taylor series with $\mathsf{RoC} = 7$. Yet

$$\mathcal{F}(y_k) = 0 = \mathcal{G}(y_k)\,, \quad \text{for each posint } k,$$

where $y_k := 7 + \frac{1}{2\pi k}$. □

*Proof of* (20). Subracting PSes gives us a PS

$$f(z) \;:=\; \sum_{n=0}^{\infty} c_n\cdot[z-P]^n$$

so that $f\lfloor_Y \equiv 0$, making $\boxed{\vec{\mathbf{c}} \stackrel{?}{=} \vec{\mathbf{0}}}$ our goal.

For each $q \in B := \mathsf{BoC}(\vec{\mathbf{c}})$, let $U(q)$ denote the *largest* centered-at-$q$ open ball that fits inside $B$. By the New-center thm, the Taylor-series for $f$, centered at $q$, converges to $f$ on all of $U(q)$.

Pick a $Y$-cluster-point $Q_0 \in B$. By (19), $f$ is identically zero on $U(Q_0)$.

On the line-segment running between $Q_0$ and $P$, we can pick a (finite) list of points

$$Q_0, Q_1, \ldots, Q_{K-1}, Q_K := P\,,$$

such that each $Q_k \in U(Q_{k-1})$. Arguing inductively, since $f$ is identically zero on $U(Q_{k-1})$, the the Taylor-series at $Q_k$ has all-zero coeffs. This therefore holds at $P$. So $\vec{\mathbf{c}} = (0,0,0,\ldots)$. ♦

21: Coro. *Suppose* $\mathcal{F}$ *and* $\mathcal{G}$ *are analytic functions on some <u>connected</u> open set* $V \subset \mathbb{C}$*. If*

$$\{z \in V \mid \mathcal{F}(z) = \mathcal{G}(z)\}$$

*has a cluster point <u>in</u>* $V$*, then* $\mathcal{F} = \mathcal{G}$*.* ♢

## Abel's theorem

We state this for a PS centered at $Q \in \mathbb{C}$.

22: Abel's thm. *With* $\mathbf{e}_n \in \mathbb{C}$, *consider power-series*

$$f(z) := \sum_{n=0}^{\infty} \mathbf{e}_n \cdot [z - Q]^n$$

*with* RoC $\mathsf{RoC} \in (0, \infty)$, *as well as a point* $z_0$ *on the circle-of-convergence,* $|z_0 - Q| = \mathsf{RoC}$. *If* $f(z_0)$ *is finite, then*

$$\lim_{t \nearrow 1} f\Big(t \cdot z_0 + [1-t]Q\Big) = f(z_0).$$

*I.e,* $f$ *is continuous along radial lines out to points-of-convergence on the boundary of* BoC. ♢

*Reduction.* WLOG $Q = 0$. WLOG $\mathsf{RoC} = 1$ and $z_0 = 1$. So

22a: $$f(x) = \sum_{n=0}^{\infty} \mathbf{e}_n \cdot x^n$$

with $\mathsf{RoC}(\vec{\mathbf{e}}) = 1$, and $\sum_{n=0}^{\infty} \mathbf{e}_n$ finite. Subtract a constant from $\mathbf{e}_0$ so that, WLOG,

22b: $$0 = f(1) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \mathbf{e}_n \stackrel{\text{def}}{=} \lim_{\ell \nearrow \infty} s_\ell,$$

where $s_\ell := \sum_{n \in [0..\ell]} \mathbf{e}_n$ is the $\ell^{th}$***-partial-sum***. Having fixed $\varepsilon > 0$, we will show

$$\forall x \in [\beta, 1): \quad |f(x)| \le 2\varepsilon,$$

for a posreal $\beta < 1$ that we will produce. □

*Pf.* Consider points $x \in J := [0, 1)$. Limit (22b) says we can take $\mathbf{L}$ so large that

$$\forall \ell \ge \mathbf{L}: \quad |s_\ell| \le \varepsilon.$$

And $\sum_{\ell=\mathbf{L}}^{\infty} s_\ell x^\ell$ equals $x^{\mathbf{L}} \cdot \sum_{n=0}^{\infty} s_{n+\mathbf{L}} x^n$. So

$$\left|[1-x] \cdot \sum_{\ell=\mathbf{L}}^{\infty} s_\ell x^\ell\right| \le x^{\mathbf{L}} \cdot [1-x] \cdot \sum_{n=0}^{\infty} \varepsilon x^n = x^{\mathbf{L}} \cdot \varepsilon \le \varepsilon,$$

since $|x| < 1$. Let's summarize.

22c: $$\forall x \in J: \quad \left|[1-x] \cdot \sum_{\ell \in [\mathbf{L}..\infty)} s_\ell x^\ell\right| \le \varepsilon.$$

**Defining** $\beta$. Wanting $[1-\beta] \cdot \sum_{\ell \in [0..\mathbf{L})} |s_\ell| < \varepsilon$, simply fix a $\beta < 1$ sufficiently close to 1. Thus

22d: $$\forall x \in [\beta, 1): \quad \left|[1-x] \cdot \sum_{\ell \in [0..\mathbf{L})} s_\ell x^\ell\right| \le \varepsilon.$$

**Absolute convergence.** At a point $x \in J$, series (22a) and $\sum_{j=0}^{\infty} x^j = \frac{1}{1-x}$ are each *absolutely convergent*. So we can multiply the series and regroup to conclude:

$$\frac{1}{1-x} \cdot f(x) = \Big[\sum_{j=0}^{\infty} 1 \cdot x^j\Big] \cdot \Big[\sum_{k=0}^{\infty} \mathbf{e}_k x^k\Big] = \sum_{\ell=0}^{\infty} \Big[\sum_{j+k=\ell} 1 \cdot \mathbf{e}_k\Big] x^\ell \stackrel{\text{note}}{=} \sum_{\ell=0}^{\infty} s_\ell x^\ell.$$

For each $x \in [\beta, 1)$, then, (22d) and (22c) give

$$|f(x)| = \left|[1-x] \cdot \sum_{\ell=0}^{\infty} s_\ell x^\ell\right| \le \varepsilon + \varepsilon. \quad ♦$$

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