A canonical structure theorem for finite joining-rank maps

Jonathan L. King* and Jean-Paul Thouvenot†

Abstract. A numerical isomorphism invariant, ***joining-rank***, was introduced in [**1**] as a quantitative generalization of Rudolph's property of minimal self-joinings. Therein, a structure theory was developed for those transformations T whose joining-rank, $\mathrm{jr}(T)$, is finite. Here, we sharpen the theorem and show it to be canonical: If $\mathrm{jr}(T) < \infty$ then there is a unique triple $\langle e, p, S \rangle$, where e and p are natural numbers and S is a map with minimal self-joinings, such that T is an e-point extension of S^p. Furthermore, the product $e \cdot p$ equals the joining-rank of T.

This theorem applies to any finite-rank mixing map, since for such maps the rank dominates the joining-rank. Another corollary is that any rank-1 transformation which is partial-mixing has minimal self-joinings. This partially answers a question of [**3**].

§I Introduction

Hints of structure theories for various classes of (typically, zero-entropy) transformations have been surfacing in the past decade and a half: Among these are the Veech and del Junco–Rudolph notion of simple maps; the weak-closure dichotomy, [**2**], for rank-1 maps; and the powerful relative compact/weak-mixing structure theorems developed by Zimmer, [**Z1**] and [**Z2**], and by Furstenberg [**F1**]; the latter gave rise to the ergodic-theoretic proof of Szemeredi's Theorem.

A previous paper *Joining-rank and the structure of finite rank mixing transformations*, [**1**], introduced the joining-rank invariant, showed that $\mathrm{jr}(T) \le \mathrm{rank}(T)$ for mixing T, and described the structure of such maps and of their commutants. We have repeated here the salient definitions and results so that the present article can be enjoyed on its own. In order to understand the motivation and details, however, its predecessor is prerequisite. Since that paper has six sections, after the introduction we number the sections of the current article §7 and §8. A reference to "theorem 2.4", then, refers to a theorem of [**1**].

Acknowledgement. The right combination of ideas to bring this work to fruition occurred in the stimulating environment of the Ergodic Theory Conference of the *Special Year in Dynamical Systems* at the University of Maryland in October 1986. Among the people whose ideas and encouragement played a role in [**1**] or in this sequel are Joe Auslander, Ken Berg, Andrés del Junco, Nelson Markley, Dan Rudolph and Benjamin Weiss.

Notation. Use the symbol := to mean "is defined to be"; thus $a := b$ means that the expression b defines the (new) symbol a. Let the expression $[n \mathinner{..} m)$ denote the half-open "interval of integers" $[n, m) \cap \mathbb{Z}$, and analogously for $[n \mathinner{..} m]$ etc. For a transformation, let $T^{\times n}$ denote the cartesian nth power $T \times \cdots \times T$; use the same convention for measures, $\mu^{\times n}$, and spaces, $X^{\times n}$.

*Department of Mathematics, University of Florida, Gainesville. Email: `squash@math.ufl.edu`
Web: `http://www.math.ufl.edu/~squash/`
Partially supported by a National Science Foundation Postdoctoral Research Fellowship.
†Université Pierre et Marie Curie, place Jussieu, Paris. Member CNRS.

Journal d'Analyse Mathématiques, vol. 56, (1991), 211–230.

Transformations. We will use ***field*** to mean σ-algebra. A measure space is a triple $(X, \mathcal{X}, \mu)$ where X is the set, $\mathcal{X}$ is the field (a scriptfont version of the symbol for the space), and μ is a Lebesgue probability measure. To indicate that T maps X to itself and preserves μ we may write $(T{:}\, X, \mu)$ or simply $(T{:}\, \mu)$. A probability measure ξ on $(Y, \mathcal{Y})$ and a measurable map $f{:}\, Y \to Y$ give rise to a new probability measure, $\xi \circ f^{-1}$, on space $(Y, \mathcal{Y})$. Use the symbol $[\![f]\!]\xi$ to denote this new measure; this emphasizes the view that f *acts* on ξ.

The ***commutant*** of T, written $C(T)$, denotes the semigroup of transformations $(S{:}\, \mu)$ such that ST=TS. For the transformations T we consider it will turn out that $C(T)$ is always a group. For an invertible T, the ***essential commutant*** $\mathrm{EC}(T)$ is the quotient (semi)group $C(T)/\{T^n\}_{n\in\mathbb{Z}}$.

Subfields. The ***trivial*** subfield is $\{\varnothing, X\}$ whereas $\mathcal{F}$ is a ***proper*** subfield if $\mathcal{F} \subsetneqq \mathcal{X}$. For a transformation $(T{:}\, X, \mu)$ a ***factor field*** $\mathcal{F}$ is a T-invariant subfield. T is ***prime*** if it has no proper non-trivial factor fields. A factor field $\mathcal{F}$ will also be called "prime" if $T|_{\mathcal{F}}$ is prime. Note: For typographical reasons, in place of $T|_{\mathcal{F}}$ we will sometimes write $T\lfloor_{\mathcal{F}}$.

Given two subfields $\mathcal{F}_1$ and $\mathcal{F}_2$ of a probability space $(X, \mathcal{X}, \mu)$, if it becomes necessary to emphasize the measure when indicating containment, agree to write $\mathcal{F}_1 \subset^\mu \mathcal{F}_2$. This means that for each $A_1 \in \mathcal{F}_1$ there exists an $A_2 \in \mathcal{F}_2$ such that $\mu(A_1 \triangle A_2) = 0$. Equality $\mathcal{F}_1 =^\mu \mathcal{F}_2$ means $\mathcal{F}_1 \subset^\mu \mathcal{F}_2$ and $\mathcal{F}_1 \supset^\mu \mathcal{F}_2$.

Conversely, let $\mathcal{F}_1 \perp^\mu \mathcal{F}_2$ indicate independence of subfields: $\mu(A_1 \cap A_2) = \mu(A_1)\mu(A_2)$ for all $A_i \in \mathcal{F}_i$.

Finally, for a collection $\{\mathcal{F}_n\}_n$ of subfields, let $\bigvee_n \mathcal{F}_n$ denote their ***join***, the smallest field containing them all.

Joinings. Fix an $N \in \mathbb{N} \cup \{\infty\}$. Given a countable list $\{(X_n, \mu_n) \mid 0 \le n < N\}$ of probability spaces let $(Y, \mathcal{Y})$ denote the cartesian product space $(\bigotimes_n X_n, \bigotimes_n \mathcal{X}_n)$. It will be convenient to view the $\mathcal{X}_n$ also as *subfields* of $\mathcal{Y}$. For instance, we may choose to write $\mathcal{Y} = \bigvee_n \mathcal{X}_n$; this convention will be employed without announcement in the sequel. Now fix a probability measure ξ on $(Y, \mathcal{Y})$. Each index-subset $I \subset [0\,..\,N)$ determines a subfield $\mathcal{F} := \bigotimes_{n\in I} \mathcal{X}_n$ and hence a ***marginal***: Written $\xi|_{\mathcal{F}}$ or $\xi|_I$, it is a probability measure on $(\bigotimes_{n\in I} X_n, \bigotimes_{n\in I} \mathcal{X}_n)$ defined on rectangles, $A_n \in \mathcal{X}_n$ for $n \in I$, by

$$[\xi|_I](\bigotimes_{n\in I} A_n) := \xi(\bigotimes_n A'_n)$$

where A'_n denotes A_n or X_n depending on whether $n \in I$ or not.

Letting $K := \#I$ denote the cardinality of I, call $\xi|_I$ a ***K-fold*** or ***K-dimensional*** marginal.

Such a measure ξ is called a ***joining of the measures*** $\mu_0, \mu_1, \ldots$ if all of its 1-dimensional marginals are "correct" ie., $\xi\lfloor_{\mathcal{X}_n} = \mu_n$, for each n. The set of such joinings we shall write as $\mathbb{M}(\mu_0, \mu_1, \ldots)$. A marginal $\xi|_I$ is a joining in $\mathbb{M}(\{\mu_n \mid n \in I\})$.

Notation for Copies. Agree to employ a subscript (or occasionally a superscript) between angle brackets to indicate a ***copy*** of a measure, field, transformation, or function. Given a space $(X, \mathcal{X}, \mu)$ we might refer to a joining $\eta \in \mathbb{M}(\mu_{\langle 1\rangle}, \mu_{\langle 2\rangle})$. Each $\mu_{\langle i\rangle}$ denotes an isomorphic copy of μ. Measure η lives on space $X_{\langle 1\rangle} \times X_{\langle 2\rangle}$ with field $\mathcal{X}_{\langle 1\rangle} \vee \mathcal{X}_{\langle 2\rangle}$.

Relative Independent Joining. Given a space $(W, \mathcal{W}, \xi)$ and a subfield $\mathcal{Y} \subset \mathcal{W}$ one can decompose ξ into a family of ***fiber measures*** over $\mathcal{Y}$. Letting $Y := W/\mathcal{Y}$ denote the quotient space and $\nu := \xi|_{\mathcal{Y}}$ the restriction measure, the fiber measures are Lebesgue probability measures

$\{\varphi_y\}_{y\in Y}$ on $(W,\mathcal{W})$ defined uniquely (mod ν) by

$$\forall B\in\mathcal{W},\, Y'\in\mathcal{Y}: \qquad \int_{Y'}\varphi_y(B)\,d\nu(y)=\xi(B\cap Y'). \tag{I.1}$$

Thus $y\mapsto\varphi_y(B)$ is simply $\mathbb{E}_{\mathcal{Y}}(B)$, the conditional expectation of the indicator function 1_B with respect to $\mathcal{Y}$.

Take R copies of ξ and define a joining $\rho\in\mathbb{M}(\xi_{\langle 1\rangle},\dots,\xi_{\langle R\rangle})$, the R-fold ***relative independent joining*** of ξ over $\mathcal{Y}$, by

$$\rho(B_1\times\cdots\times B_R):=\int_Y\varphi_y(B_1)\cdot\varphi_y(B_2)\cdot\ldots\cdot\varphi_y(B_R)\,d\nu(y), \qquad B_r\in\mathcal{W}. \tag{I.2}$$

In the case $R=2$ we may write ρ as $\xi_{\langle 1\rangle}\times_{\mathcal{Y}}\xi_{\langle 2\rangle}$ or just $\xi\times_{\mathcal{Y}}\xi$.

Viewing $\mathcal{Y},\mathcal{W}_{\langle 1\rangle},\mathcal{W}_{\langle 2\rangle}$ as subfields in $\rho:=\xi_{\langle 1\rangle}\times_{\mathcal{Y}}\xi_{\langle 2\rangle}$, one can check that $\mathcal{W}_{\langle 1\rangle}$ and $\mathcal{W}_{\langle 2\rangle}$ are conditionally independent relative to $\mathcal{Y}$. This means

For bounded $\mathcal{W}_{\langle i\rangle}$-measurable functions f_i on $W_{\langle 1\rangle}\times W_{\langle 2\rangle}$: $\quad \mathbb{E}_{\mathcal{Y}}(f_1\cdot f_2)=\mathbb{E}_{\mathcal{Y}}(f_1)\cdot\mathbb{E}_{\mathcal{Y}}(f_2)$

where the conditional expectations are taken relative to the measure ρ. Note for later convenience the following special case:

$$\textit{If } B\in\mathcal{W} \textit{ satisfies } \big[\xi\times_{\mathcal{Y}}\xi\big](B\times B)=\xi(B) \textit{ then } B\in\mathcal{Y}. \tag{I.3}$$

The equality says $\int[\varphi_y(B)]^2\,d\nu(y)=\int\varphi_y(B)\,d\nu(y)$, where $0\le\varphi_y(B)\le 1$. Since a number strictly between 0 and 1 strictly dominates its square, equality between the integrals can occur only if $\varphi_y(B)\in\{0,1\}$ for a.e. y. Thus $B\in\mathcal{Y}$. $\Diamond$

In applications, the measure ξ above will itself be a joining. Part *(a)* below is an easy consequence of conditional independence. Part *(b)* is the condition for equality of Jensen's Inequality in the language of joinings, a simple but important observation by del Junco and Rudolph.

LEMMA I.4. *Given (X,μ) and (Y,ν) and a joining $\xi\in\mathbb{M}(\mu,\nu)$, let $\rho:=\xi_{\langle 1\rangle}\times_{\mathcal{Y}}\xi_{\langle 2\rangle}$ be the relative independent joining over Y.*

(a) *If $\mathcal{X}_{\langle 1\rangle}\stackrel{\rho}{=}\mathcal{X}_{\langle 2\rangle}$ then $\mathcal{Y}\supset^{\xi}\mathcal{X}$.*

(b) *If $\mathcal{X}_{\langle 1\rangle}\perp^{\rho}\mathcal{X}_{\langle 2\rangle}$ then $\mathcal{Y}\perp^{\xi}\mathcal{X}$.*

PROOF OF *(b)*: Let $A_1,A_2,\dots\in\mathcal{X}$ be a countable collection of sets which separates the points of X. Then any Lebesgue measure φ on $(X,\mathcal{X})$ is determined by its list $\varphi(A_1),\varphi(A_2),\dots$ of values.

Let $\{\varphi_y\}_{y\in Y}$ be the fiber measure decomposition of ξ over $\mathcal{Y}$. View the fiber measures as living on $(X,\mathcal{X})$. In this notation

$$\mu(A)=\xi(A\times Y)=\int_Y\varphi_y(A)\,d\nu(y) \tag{I.5}$$

for each $A\in\mathcal{X}$. By the hypothesis of independence,

$$\left[\int\varphi_y(A)\,d\nu(y)\right]^2=\mu(A)\mu(A)\stackrel{\text{indep.}}{=\!=}\rho|_{\mathcal{X}_{\langle 1\rangle}\vee\mathcal{X}_{\langle 2\rangle}}(A\times A)$$
$$=\int\big[\varphi_y(A)\big]^2\,d\nu(y).$$

Thus we have equality in Jensen's Inequality and so $y \mapsto \varphi_y(A)$ is constant for all y outside of some ν-nullset $\mathcal{N}(A)$. Consequently, $y \mapsto \varphi_y$ is constant for y off the nullset $\bigcup_{n=1}^{\infty} \mathcal{N}(A_n)$. This constant must be μ since the marginal $\xi|_{\mathcal{X}}$ is μ. Hence $\xi = \mu \times \nu$. ◊

Transformation joinings. Suppose $(S\colon W, \lambda)$ and $(T\colon X, \mu)$ are transformations. Given a joining ξ of measures λ and μ, one can let $S \times T$ act on ξ to create a new joining $[\![S \times T]\!]\xi$ in $\mathbb{M}(\lambda, \mu)$.

$$\big[[\![S \times T]\!]\xi\big](A \times B) := \xi\big(S^{-1}(A) \times T^{-1}(B)\big) \qquad \text{for } A \in \mathcal{W}, B \in \mathcal{X}.$$

A $\xi \in \mathbb{M}(\lambda, \mu)$ is called a ***joining of the transformations*** S ***and*** T if it is $S{\times}T$ invariant, that is, if $[\![S{\times}T]\!]\xi = \xi$. Use $\mathbb{J}(S, T)$ to denote the set of such joinings. Saying that a joining ξ is "ergodic" or "weak-mixing" means that the transformation $(S{\times}T\colon \xi)$ is. Let $\mathbb{J}_{\mathrm{Erg}}(S, T)$ denote the ergodic members of $\mathbb{J}(S, T)$.

The relative independent joining over a common factor provides an example of a joining of transformations. Suppose $\mathcal{F} \subset \mathcal{W}$ and $\mathcal{G} \subset \mathcal{X}$ are factor fields of S and T such that

$$S|_{\mathcal{F}} \cong T|_{\mathcal{G}}.$$

Make the isomorphism implicit and denote this factor transformation as $(U\colon Y, \mathcal{Y}, \nu)$ by making the identifications $W/\mathcal{F} = Y = X/\mathcal{G}$ and $\lambda|_{\mathcal{F}} = \nu = \mu|_{\mathcal{G}}$. With respect to the factor maps

$$(S\colon \lambda) \longrightarrow (U\colon \nu) \longleftarrow (T\colon \mu) \tag{I.6a}$$

the relative independent joining $\rho \in \mathbb{J}(S, T)$, defined for $A \in \mathcal{W}$ and $B \in \mathcal{X}$, is

$$\rho(A \times B) := \int_Y \mathbb{E}^{\lambda}_{\mathcal{F}}(A) \cdot \mathbb{E}^{\mu}_{\mathcal{G}}(B)\, d\nu \tag{I.6b}$$

where the conditional expectations are taken relative to the superscript measures.

A member $\xi \in \mathbb{J}(T, T)$ is called a (2-fold) ***self-joining*** of T. An important example is the ***diagonal joining*** Δ_μ defined by $\Delta_\mu(A \times B) := \mu(A \cap B)$. Each S in the commutant of T gives rise to a ***graph joining*** $[\![I \times S]\!]\Delta_\mu$, where I is the identity transformation. In the case where S is some power T^n, this joining will be called the n-th ***off-diagonal*** joining. Evidently in a graph joining $\xi := [\![I \times S]\!]\Delta_\mu$ the two subfields are identified ie., $\mathcal{X}_{\langle 1 \rangle} \stackrel{\xi}{=} \mathcal{X}_{\langle 2 \rangle}$. The converse,

$$\begin{array}{l}\text{If } \xi \in \mathbb{J}(T_{\langle 1 \rangle}, T_{\langle 2 \rangle}) \text{ and } \mathcal{X}_{\langle 1 \rangle} \stackrel{\xi}{=} \mathcal{X}_{\langle 2 \rangle} \text{ then, for} \\ \text{some invertible } S \in C(T)\text{:} \quad \xi = [\![I \times S]\!]\Delta_\mu\end{array} \tag{I.7}$$

can be easily verified.

Let Δ^M_μ denote the M-fold diagonal self-joining whose value $\Delta^M_\mu(A_1 {\times} \ldots {\times} A_M)$ equals $\mu(A_1 \cap \cdots \cap A_M)$, each $A_m \in \mathcal{X}$. Given $(T\colon \mu)$ we may occasionally use the measure μ as a synonym for T, when T is the only transformation acting on it. Thus we might refer to the commutant $C(\mu)$. This will usually be done when μ is itself a joining and we have not bothered to give a name to the transformation acting on it. Particularly common, given $(T\colon \mu)$ and $(S\colon \lambda)$, will be to use "$\mathbb{J}(\mu, \lambda)$" to mean $\mathbb{J}(T, S)$.

DEFINITION. Say that a 2-fold self-joining of T is ***trivial*** if is either product measure $\mu \times \mu$ or is an off-diagonal joining, $[\![I{\times}T^n]\!]\Delta_\mu$. Transformation T has (2-fold) ***minimal self-joinings*** if $\mathbb{J}_{\mathrm{Erg}}(T, T)$ contains only trivial self-joinings, [**R**] and [**J,R**]. For r copies of T, let the set of

pairwise non-trivial joinings be denoted by $\mathbb{J}_{\mathrm{PNT}}(T,\ldots,T)$. These are the ergodic r-fold self-joinings such that each 2-dimensional marginal is non-trivial.

The ***joining-rank*** of T, written $\mathrm{jr}(T)$, is the supremum of natural numbers R such that $\mathbb{J}_{\mathrm{PNT}}(\{T\}_1^R)$ is non-empty. Thus "joining-rank 1" is equivalent to minimal self-joinings. Theorem 2.4 asserts that, other than the trivial case of a discrete rotation, $\mathrm{jr}(T) < \infty$ implies T is weak-mixing. Courtesy of this, henceforth

All maps of finite joining-rank are presumed weak-mixing.

That the class of such maps is large, is shown by (I.8) as well as (7.1) and the remark following it.

Extensions. Choose a transformation $(S{:}\,W,\lambda)$, a space (Y,ν) and a measurable family $\{U_w \mid w \in W\}$ of ν-preserving transformations on Y. An ***extension*** T, of S, is a $\lambda\times\nu$ preserving transformation of $W \times Y$ defined by

$$w, y \overset{T}{\longmapsto} Sw, U_w(y).$$

If Y is a finite set and ν assigns mass $1/n$ to each point $y \in Y$, where $n := \#Y$, then T is called an ***n-point extension*** (or just "finite extension") of S.

Given $(T{:}\,X,\mu)$ ergodic and a factor field $\mathcal{W} \subset \mathcal{X}$, set $S := T|_{\mathcal{W}}$. Then T is an extension of S. We say this extension is ***canonical*** if $T|_{\mathcal{W}'} \cong T|_{\mathcal{W}}$ implies $\mathcal{W}' = \mathcal{W}$. This is stronger than just saying that $\mathcal{W}$ is fixed by all automorphisms of T.

Peripheral terms. Recall that T is ***partial mixing*** if $\alpha(T) > 0$, where $\alpha(T)$ is the supremum of all $\alpha \in [0,1]$ such that $\liminf_{n\to\infty} \mu(A \cap T^{-n}B) \geq \alpha \cdot \mu(A)\mu(B)$ for all $A, B \in \mathcal{X}$.

The reader unfamiliar with the following notions, which play only a minor role in the sequel, is referred to §1: The ***rank*** of T, $\mathrm{rk}(T)$, a value in $[1\mathinner{..}\infty]$. The ***covering number*** of T, written $\kappa(T)$; always $0 \leq \kappa(T) \leq 1$ and $1/\mathrm{rk}(T) \leq \kappa(T)$. Map T is ***rigid*** if there exists $n_k \nearrow \infty$ such that $\mu(A \cap T^{-n_k}A) \to \mu(A)$ for all $A \in \mathcal{X}$.

Some notation has changed: We use $\mathbb{J}(S,T)$ for what was called "Joi(S,T)" in [**1**] and $\mathrm{jr}(\cdot)$ for "jrk$(\cdot)$". Also, $[\![S \times T]\!]\xi$ replaces "$(S \times T)\xi$" and $[\![I \times T^n]\!]\Delta_\mu$ is used instead of "Δ_μ^n".

The below is collated from theorems 2.4, 2.7, 2.9 and 3.5.

THEOREM I.8. *The class of finite joining-rank transformations is closed under roots, powers, factors, and weak-mixing finite extensions. Let R denote the joining-rank of T.*

(a) *For any $n \geq 1$: $\mathrm{jr}(T^n) = n \cdot \mathrm{jr}(T)$. Also, $\#\mathrm{EC}(T) \leq R$ and $C(T)$ is a group. Indeed, any $S \in C(T)$ is a root, at most an Rth root, of some power of T.*

(b) *If S is a weak-mixing n-point extension of T then $\mathrm{jr}(S) = n \cdot \mathrm{jr}(T)$. Over any of its non-trivial factors T is a finite, at most R-point, extension.*

(c) *T mixing: $\mathrm{jr}(T) \leq \mathrm{rk}(T)$.*

(d) *T rank-1: $\mathrm{jr}(T) \leq 1/\alpha(T)$.*

RUDIMENTARY STRUCTURE THEOREM, 4.9. *Suppose T is of finite joining-rank. Then there exists a factor field $\mathcal{P} \subset \mathcal{X}$ such that $T|_{\mathcal{P}}$ is prime and this factor transformation has a root S which is prime and has trivial commutant. Thus T is an e-point extension of the p-th power of S, where e and p are natural numbers satisfying*

$$\mathrm{jr}(T) = e \cdot p \cdot \mathrm{jr}(S).$$

If T has only one prime factor field then the above description is canonical and the numbers e and p as well as the transformation S, are unique.

§7 Canonical Structure

The goal of this section is to strengthen the above theorem to the following.

Structure Theorem. *Suppose* $\mathrm{jr}(T) < \infty$*. Then* T *is an* e*-point extension of the* p*-th power of a map* S *with minimal self-joinings, and* $\mathrm{jr}(T) = e \cdot p$*. Furthermore, the extension is canonical and* e*,* p *and* S *are unique.*

An unexpected application of this theorem provides a large class of maps with minimal self-joinings. This provides some slight evidence for the joining-closure conjecture (4.1) of [**3**].

Theorem 7.1. *Any partial mixing rank-1* T *has minimal self-joinings.*

Proof: T has joining-rank bounded by $1/\alpha(T)$ and so T is an e-point extension of some S^p as above. By the Weak-closure Theorem of [**2**] any proper factor of a rank-1 must be rigid. But "partial mixing" is closed under factors and is incompatible with rigidity; thus $e = 1$ and so $T = S^p$.

Since T is partial mixing it is non-rigid and so the Weak-closure Theorem asserts that T only commutes with its powers and consequently has no roots. Thus $T = S$ and hence has minimal self-joinings. ◊

Remark. There is a weak converse to this. A rank-1 map is one whose covering number $\kappa(T)$ equals 1. Now theorem 3.5 asserts

$$\mathrm{jr}(T) \leq \left\lfloor \frac{1}{\kappa(T) + \alpha(T) - 1} \right\rfloor .$$

We have seen that for a partial mixing map, if $\kappa(T) = 1$ then it has minimal self-joinings. On the other hand, given ε it is possible to build a partial mixing T as a cartesian product $S_1 \times S_2$ of non-trivial maps, such that $\kappa(T) > 1 - \varepsilon$. Since T is a cartesian product, though, its joining-rank is infinite by (I.8b).

Lifting product measure. A notion of relative disjointness of an extension is needed. Fix an ergodic $(T\colon X, \mathcal{X}, \mu)$. Given factors $\mathcal{F} \subset \mathcal{G} \subset \mathcal{X}$ we say that $\mathcal{G}$ is a ***(product) lift*** of $\mathcal{F}$ if

$$\forall \xi \in \mathrm{J}_{\mathrm{Erg}}(T, S) : \qquad \mathcal{F} \perp^{\xi} \mathcal{W} \implies \mathcal{G} \perp^{\xi} \mathcal{W}$$

whenever $(S\colon W, \lambda)$ is a weak-mixing transformation. With respect to μ, say that $\mathcal{G}$ is a ***maximal lift*** of $\mathcal{F}$ if whenever $\mathcal{G}' \supset \mathcal{G}$ is a product lift of $\mathcal{F}$ then $\mathcal{G}' = \mathcal{G}$. Use the following containment symbols

$$\begin{array}{ll} \mathcal{G} \sqsupset^{\mu} \mathcal{F} & \mathcal{G} \text{ a lift of } \mathcal{F} \\ \mathcal{G} \sqsupseteq^{\mu} \mathcal{F} & \mathcal{G} \text{ a maximal lift of } \mathcal{F} \end{array}$$

to denote these relations between subfields of $\mathcal{X}$.

An ***increasing tower of fields*** in $\mathcal{X}$ is a transfinite sequence $\{\mathcal{G}_\alpha \mid \alpha < \beta\}$ where β is an ordinal, α ranges over all ordinals less than β, and $\mathcal{G}_\alpha \subset \mathcal{G}_{\alpha'}$ whenever $\alpha < \alpha'$.

Lemma 7.2. *Consider factors fields* $\mathcal{F}, \mathcal{G}, \mathcal{H} \subset \mathcal{X}$*.*

(a) *If* $\mathcal{H} \sqsupset^{\mu} \mathcal{G}$ *and* $\mathcal{G} \sqsupset^{\mu} \mathcal{F}$ *then* $\mathcal{H} \sqsupset^{\mu} \mathcal{F}$*.*

(b) *If* $\{\mathcal{G}_\alpha \mid \alpha < \beta\}$ *is a tower of factors with* $\mathcal{F} \subset \mathcal{G}_0$ *then*

$$\left[\forall\alpha:\ \mathcal{G}_\alpha \sqsupset^\mu \mathcal{F}\right] \implies \bigvee_\alpha \mathcal{G}_\alpha \sqsupset^\mu \mathcal{F}.$$

Consequently, by Zorn's lemma, any factor $\mathcal{F} \subset \mathcal{X}$ *has a maximal* μ*-lift.*

Proof of *(b):* Let $\mathcal{H}$ denote $\bigvee_\alpha \mathcal{G}_\alpha$. Since our space is Lebesgue there exists $\mathcal{D} \subset \mathcal{H}$, a countable dense subcollection, which can be listed $A_1, A_2, \ldots$ so that every member of $\mathcal{D}$ occurs infinitely often. Choose an increasing sequence $\{\alpha_n\}_{n=1}^\infty$ of ordinals such that there exists a set $B_n \in \mathcal{G}_{\alpha_n}$ with $\mu(B_n \triangle A_n) < 1/n$. This shows that there is a countable subtower such that

$$\bigvee_{n=1}^{\infty} \mathcal{G}_{\alpha_n} = \mathcal{H}.$$

and so we can write n for α_n and say $\mathcal{G}_n \nearrow \mathcal{H}$.

Fix a weak-mixing $(S\colon W)$ and $\xi \in \mathbb{J}_{\mathrm{Erg}}(T,S)$; view $\mathcal{G}_n, \mathcal{H}$ and $\mathcal{W}$ as subfields of $\mathcal{X} \times \mathcal{W}$. Taking an $\mathcal{H}$-measurable function h and $\mathcal{W}$-measurable f, both bounded, we wish to show that $\mathbb{E}(h\cdot f) = \mathbb{E}(h)\cdot\mathbb{E}(f)$ where $\mathbb{E}$ denotes conditional expectation with respect to the joining ξ. But by hypothesis the equality holds for h replaced by each conditional expectation $g_n := \mathbb{E}_{\mathcal{G}_n}(h)$. And $g_n \to h$ in L^1 by the Martingale Convergence Theorem. ◊

Algebraic Extensions. The introduction discussed the notion of a finite extension. A well-known generalization of this is an ***isometric extension*** whose definition appears in the Appendix along with a proof of the following theorem.

Theorem 7.3. *Any ergodic isometric extension of a transformation* U *is a product lift of* U.

Part (a) below is Proposition 6.4 of Furstenberg's book [**F1**]. Part (*b*) follows from Zimmer's papers [**Z1**] and [**Z2**], assertions 4.3, 7.3 and 7.10. (In a slightly different formulation, (*b*) also appears in Furstenberg's book.)

Theorem 7.4. $(T\colon X,\mu)$ *ergodic and* $\mathcal{G} \subset \mathcal{X}$ *a factor.*

(a) *If* $\mu \times_{\mathcal{G}} \mu$ *is ergodic then for all* $M \in \mathbb{N}$ *the* M*-fold relative independent joining over* $\mathcal{G}$ *is ergodic.*

(b) *If* $\mu \times_{\mathcal{G}} \mu$ *fails to be ergodic then there exists* $\mathcal{H} \supsetneqq \mathcal{G}$*, a non-trivial isometric extension of* $\mathcal{G}$*, inside of* $\mathcal{X}$.

Part *(b)* and the theorem before combine to yield the following.

Lifting Lemma. *If* $\mathcal{G} \sqsupseteq^\mu \mathcal{F}$ *then* $\mu \times_{\mathcal{G}} \mu$ *is ergodic.*

The maximal-joining

For any M and self-joining $\eta \in \mathbb{J}(\{T\}_1^M)$ say that a non-trivial factor field $\mathcal{F} \subset \bigvee_{m=1}^M \mathcal{X}_{\langle m\rangle}$ is a ***component-factor*** in η if $\mathcal{F} \subset^\eta \mathcal{X}_{\langle m\rangle}$ for some m.

Main Lemma 7.5. *Suppose* $R := \mathrm{jr}(T)$ *is finite.*

(*i*) *Given any* $(S\colon W,\lambda)$ *and* $\xi \in \mathbb{J}_{\mathrm{Erg}}(S,T)$ *with* $\mathcal{W} \not\perp^\xi \mathcal{X}$. *Then* $\mathcal{W} \vee \mathcal{X} \sqsupset^\xi \mathcal{W}$.

(*ii*) *Given a pairwise non-trivial joining* $\eta \in \mathbb{J}_{\mathrm{PNT}}(\{T\}_1^M)$ *and any component-factor* $\mathcal{F}$ *in* η:

$$\mathcal{F} \sqsubset^\eta \mathcal{X}_{\langle 1\rangle} \vee \mathcal{X}_{\langle 2\rangle} \vee \cdots \vee \mathcal{X}_{\langle M\rangle}.$$

PROOF OF *(i):* Let $\mathcal{Y}$ be a maximal product lift, $\mathcal{Y} \sqsupseteq^{\xi} \mathcal{W}$, in $\mathcal{W} \vee \mathcal{X}$ and let $\beta \in \mathbb{J}(\xi_{\langle 1 \rangle}, \ldots, \xi_{\langle R+1 \rangle})$ be the $(R+1)$-fold relative independent joining of ξ over this maximal $\mathcal{Y}$. By the Lifting lemma and (7.4a) this β is ergodic and so some two of $\mathcal{X}_{\langle 1 \rangle}, \ldots, \mathcal{X}_{\langle R+1 \rangle}$ must be joined trivially. But all two-fold marginals in a relative independent joining are isomorphic. Consequently, for $\rho := \xi_{\langle 1 \rangle} \times_{\mathcal{Y}} \xi_{\langle 2 \rangle}$

$$\textit{Either } \mathcal{X}_{\langle 1 \rangle} \perp^{\rho} \mathcal{X}_{\langle 2 \rangle} \textit{ or } \mathcal{X}_{\langle 1 \rangle} =^{\rho} \mathcal{X}_{\langle 2 \rangle}.$$

By hypothesis $\mathcal{W} \not\perp^{\xi} \mathcal{X}$. Consequently $\mathcal{Y}$ fails to be ξ-independent of $\mathcal{X}$ and so, by (I.4b), $\mathcal{X}_{\langle 1 \rangle} \not\perp^{\rho} \mathcal{X}_{\langle 2 \rangle}$. Hence $\mathcal{X}_{\langle 1 \rangle} =^{\rho} \mathcal{X}_{\langle 2 \rangle}$ and so (I.4a) implies that $\mathcal{Y} \supset^{\xi} \mathcal{X}$. Thus $\mathcal{Y} =^{\xi} \mathcal{W} \vee \mathcal{X}$. ◊

PROOF OF *(ii):* Assume $\mathcal{F} \subset \mathcal{X}_{\langle 1 \rangle}$. Since $\mathcal{F}$ is non-trivial it is not independent of $\mathcal{X}_{\langle 1 \rangle}$ and so part *(i)* yields that $\mathcal{F} \sqsubset^{\eta} \mathcal{F} \vee \mathcal{X}_{\langle 1 \rangle}$. Applying it another $M-1$ times obtains

$$\mathcal{F} \sqsubset^{\eta} \mathcal{X}_{\langle 1 \rangle} \sqsubset^{\eta} \mathcal{X}_{\langle 1 \rangle} \vee \mathcal{X}_{\langle 2 \rangle} \sqsubset^{\eta} \mathcal{X}_{\langle 1 \rangle} \vee \mathcal{X}_{\langle 2 \rangle} \vee \mathcal{X}_{\langle 3 \rangle} \sqsubset^{\eta} \cdots \sqsubset^{\eta} \bigvee_{m=1}^{M} \mathcal{X}_{\langle m \rangle}.$$

Now (7.2a) implies that $\mathcal{F} \sqsubset^{\eta} \bigvee_1^M \mathcal{X}_{\langle m \rangle}$. ◊

Uniqueness. Refer to any member of $\mathbb{J}_{\mathrm{PNT}}(\{T\}_1^R)$ as a ***maximal-joining***. In what sense is a maximal-joining, call it $\breve{\mu}$, unique? A superficially "different" maximal-joining could be obtained from $\breve{\mu}$ by applying powers of T to its coordinates or by permuting its coordinates. That is, for integers $n_1 \ldots, n_R$ and a permutation π of $[1 \,..\, R]$,

$$[\![(T^{n_1} \times \cdots \times T^{n_R}) \circ \widehat{\pi}]\!] \breve{\mu}$$

is "another" maximal-joining, where $\widehat{\pi}: X^{\times R} \to X^{\times R}$ sends $(x_1, \ldots, x_R)$ to $(x_{\pi(1)}, \ldots, x_{\pi(R)})$. Our goal is to show that this is the only possibility.

Say that self-joinings $\eta, \beta \in \mathbb{J}(\{T\}_1^M)$ are ***permutation-equivalent*** if

$$\beta = [\![T^{n_1} \times \ldots \times T^{n_M} \circ \widehat{\pi}]\!] \eta$$

for some integers $n_1, \ldots, n_M$ and permutation π of $[1 \,..\, M]$. In order to show that the maximal-joining is unique upto permutation-equivalence, fix two maximal-joinings $\eta^{\mathbf{1}}$ and $\eta^{\mathbf{2}}$. For $k = \mathbf{1}, \mathbf{2}$, let $\breve{\mathcal{X}}^k := \mathcal{X}^k_{\langle 1 \rangle} \vee \cdots \vee \mathcal{X}^k_{\langle R \rangle}$ denote the join of the component fields of η^k. Consider a $\xi \in \mathbb{J}_{\mathrm{Erg}}(\eta^{\mathbf{1}}, \eta^{\mathbf{2}})$.

First case: There exists $i, j \in [1 \,..\, R]$ ***such that*** $\mathcal{X}^{\mathbf{1}}_{\langle i \rangle} \perp^{\xi} \mathcal{X}^{\mathbf{2}}_{\langle j \rangle}$. Letting $\mathcal{X}^{\mathbf{2}}_{\langle j \rangle}$ play the role of $\mathcal{F}$ in (7.5*ii*) we conclude, since T is weak-mixing, that the independence lifts to $\mathcal{X}^{\mathbf{1}}_{\langle i \rangle} \perp^{\xi} \breve{\mathcal{X}}^{\mathbf{2}}$. Reversing the roles of "**1**" and "**2**" and then again, yields

$$\forall r \in [1 \,..\, R]: \qquad \breve{\mathcal{X}}^{\mathbf{1}} \perp^{\xi} \mathcal{X}^{\mathbf{2}}_{\langle r \rangle} \text{ and } \mathcal{X}^{\mathbf{1}}_{\langle r \rangle} \perp^{\xi} \breve{\mathcal{X}}^{\mathbf{2}}. \tag{7.6}$$

(It is natural to want to lift $\mathcal{X}^{\mathbf{1}}_{\langle i \rangle} \perp^{\xi} \breve{\mathcal{X}}^{\mathbf{2}}$ to full independence $\breve{\mathcal{X}}^{\mathbf{1}} \perp^{\xi} \breve{\mathcal{X}}^{\mathbf{2}}$. However, for this one would need to know that $\eta^{\mathbf{2}}$ is weak-mixing.)

Second case: For each i, j ***pair,*** $\mathcal{X}^{\mathbf{1}}_{\langle i\rangle} \not\perp^{\xi} \mathcal{X}^{\mathbf{2}}_{\langle j\rangle}$. Now fix any i and note that

$$\xi\lfloor\mathcal{X}^{\mathbf{1}}_{\langle i\rangle} \vee (\mathcal{X}^{\mathbf{2}}_{\langle 1\rangle} \vee \cdots \vee \mathcal{X}^{\mathbf{2}}_{\langle R\rangle})$$

is an ergodic $(R+1)$-fold self-joining of T. Hence some 2-fold marginal is trivial—and it must be of the form

$$\xi\lfloor\mathcal{X}^{\mathbf{1}}_{\langle i\rangle} \vee \mathcal{X}^{\mathbf{2}}_{\langle j\rangle}$$

since $\eta^{\mathbf{2}}$ is pairwise non-trivial. By hypothesis this 2-fold marginal is not product measure, so it must be some off-diagonal $[\![I\times T^n]\!]\Delta_\mu$. Thus there are maps $\pi\colon[1\,..\,R]\to[1\,..\,R]$ and $n\colon[1\,..\,R]\to\mathbb{Z}$ with

$$\xi\lfloor\mathcal{X}^{\mathbf{1}}_{\langle i\rangle} \vee \mathcal{X}^{\mathbf{2}}_{\langle\pi(i)\rangle} = [\![I\times T^{n(i)}]\!]\Delta_\mu \tag{7.7}$$

Furthermore, if $\pi(i)=\pi(i')$ then the fields $\mathcal{X}^{\mathbf{1}}_{\langle i\rangle}$ and $\mathcal{X}^{\mathbf{1}}_{\langle i'\rangle}$ are linked in ξ by an off-diagonal—hence $i=i'$, since $\eta^{\mathbf{1}}$ is pairwise non-trivial. Thus π is injective. Hence it is a permutation, and

$$\eta^{\mathbf{2}} = [\![S]\!]\eta^{\mathbf{1}} \qquad \text{where} \quad S := T^{n(1)}\times\ldots\times T^{n(R)}\circ\widehat{\pi}. \tag{7.8}$$

Application. Identify factor $\mathcal{X}^{\mathbf{1}}_{\langle 1\rangle}$ with $\mathcal{X}^{\mathbf{2}}_{\langle 1\rangle}$, then take ξ to be an ergodic component of the relative independent joining of $\eta^{\mathbf{1}}$ with $\eta^{\mathbf{2}}$ over $\mathcal{X}^{\mathbf{1}}_{\langle 1\rangle} = \mathcal{X}^{\mathbf{2}}_{\langle 1\rangle}$. For this ξ assertion (7.6) fails and so (7.8) must hold.

Courtesy of this, the maximal-joining is "unique"; fix some choice and call it $\breve{\mu}$. Indeed, for any $(T\colon X,\mu)$ of finite joining-rank R, agree to use the symbols $\breve{\mu},\breve{X},\breve{\mathcal{X}}$ and $\breve{T}$ to denote its maximal-joining, space $\breve{X} := X^{\times R}$, field $\breve{\mathcal{X}} := \bigvee_{r=1}^{R}\mathcal{X}_{\langle r\rangle}$, and transformation $(\breve{T}\colon\breve{\mu}) := (T^{\times R}\colon\breve{\mu})$. Before collecting the foregoing results as parts *(a,b)* of the lemma below, let us glance at an example.

Illustration. Suppose $\mathrm{jr}(T)=2$ and $V\in C(T)$ is not a power of T. So $\breve{\mu} := [\![I\times V]\!]\Delta_\mu$ can represent the maximal-joining. Any $S\in C(\breve{T})$ provides an ergodic self-joining, $[\![I\times S]\!]\Delta_{\breve{\mu}}$, of $\breve{T}$. But $S := V\times V$ is in $C(\breve{T})$ and so –somehow!– S has to be as in (7.8), measure-theoretically.

Since $\#\mathrm{EC}(T)\le\mathrm{jr}(T)=2$, transformation V must be a square root of some power of T, say, $V^2=T^7$. Thus

$$(V\times V\colon\breve{\mu}) = (I\times T^7\circ\widehat{\pi}\colon\breve{\mu})$$

where π is the flip, $\pi(1)=2$ and $\pi(2)=1$.

Maximal-joining Lemma, 7.9. *Given $(T\colon\mu)$ of joining-rank R. Let $\breve{\mu}\in\mathbb{J}_{\mathrm{PNT}}(\{T\}_1^R)$ denote a maximal-joining and let $\mathcal{X}_{\langle 1\rangle},\ldots,\mathcal{X}_{\langle R\rangle}$ denote the component subfields in $\breve{\mu}$.*

(a) *This $\breve{\mu}$ is unique upto permutation-equivalence. Indeed, any pairwise non-trivial joining $\eta\in\mathbb{J}_{\mathrm{PNT}}(\{T\}_1^M)$ is permutation-equivalent to some M-dimensional marginal of $\breve{\mu}$.*

(b) *Consider a $\xi\in\mathbb{J}_{\mathrm{Erg}}(\breve{\mu}^{\langle 1\rangle},\breve{\mu}^{\langle 2\rangle})$. If some component of $\breve{\mu}^{\langle 1\rangle}$ fails to be ξ-independent of some component of $\breve{\mu}^{\langle 2\rangle}$ then ξ is a graph joining $[\![I\times S]\!]\Delta_{\breve{\mu}}$ for an $S\in C(\breve{T})$.*

Conversely, suppose some component of $\breve{\mu}^{\langle 1\rangle}$ is ξ-independent of some component of $\breve{\mu}^{\langle 2\rangle}$. If $\breve{T}$ is weak-mixing then ξ is product measure $\breve{\mu}^{\langle 1\rangle}\times\breve{\mu}^{\langle 2\rangle}$.

(c) *Any $S\in C(\breve{T})$ is of the form $T^{n(1)}\times\ldots\times T^{n(R)}\circ\widehat{\pi}$. If π is the identity permutation then S is a power of $\breve{T}$.*

(*d*) $\#\mathrm{EC}(\breve{T}) \le R!$ *is an upper bound on the essential commutant.*

PROOF OF (*a*): Of necessity $M \le R$. Apply the first part of the foregoing argument with $\eta^{\mathbf{2}} := \breve{\mu}$ and $\eta^{\mathbf{1}} := \eta$. Let $\xi \in \mathbb{J}_{\mathrm{Erg}}(\eta^{\mathbf{1}}, \eta^{\mathbf{2}})$ be an ergodic component of the relative independent joining of $\eta^{\mathbf{1}}$ with $\eta^{\mathbf{2}}$ over $\mathcal{X}_{\langle 1\rangle}$. The "Second case" argument leading to (7.8) shows that π is an injection.

PROOF OF (*b*): Let $\eta^{\mathbf{1}} := \breve{\mu}^{\langle 1\rangle}$ and $\eta^{\mathbf{2}} := \breve{\mu}^{\langle 2\rangle}$. Then ξ equals $[\![I \times S]\!]\Delta_{\breve{\mu}}$ for an S as in (7.8). This forces $S \in C(\breve{T})$, recalling (I.7).

Conversely, is $\breve{\mu}$ is weak-mixing and some $\mathcal{X}^{\mathbf{1}}_{\langle i\rangle} \perp^{\xi} \mathcal{X}^{\mathbf{2}}_{\langle j\rangle}$ then we can lift the partial independence of (7.6) to total independence $\breve{\mu}^{\langle 1\rangle} \perp^{\xi} \breve{\mu}^{\langle 2\rangle}$.

PROOF OF (*c*): Suppose now that the π corresponding to S is the identity; we may assume that S is of the form $I{\times}T^{n(2)}{\times}T^{n(3)}{\times}\ldots{\times}T^{n(R)}$. For each $k \ge 2$, the marginal measure $\breve{\mu}\lfloor\mathcal{X}_{\langle 1\rangle} \vee \mathcal{X}_{\langle k\rangle}$ is $I{\times}T^{n(k)}$ invariant. But $\mathcal{X}_{\langle 1\rangle} \not\perp^{\breve{\mu}} \mathcal{X}_{\langle k\rangle}$. As is well known, the identity map is "disjoint" from any ergodic map (see Appendix) and so $T^{n(k)}$ must fail to be ergodic; whence $n(k) = 0$.

PROOF OF (*d*): Any collection of $1+R!$ many maps in $C(\breve{T})$ must have two members, call them S and U, with the same permutation π. Hence SU^{-1} is a power $\breve{T}$. ◊

Properties of the commutant

Given transformation $(U{:}\,Y, \nu)$ and a subgroup $G \subset C(U)$ let $\mathbf{F}[G]$ denote the ***fixed field*** of G

$$\mathbf{F}[G] := \{A \in \mathcal{Y} \mid \forall S \in G,\, S^{-1}(A) = A\}.$$

This is a U-invariant subfield of $\mathcal{Y}$.

COMMUTANT LEMMA, 7.10. *Suppose $(U{:}\,Y, \nu)$ is aperiodic.*

(*a*) *If $G \subset C(U)$ is a finite subgroup then $\mathbf{F}[G]$ is a non-trivial subfield.*

(*b*) *Suppose G_1, G_2 are subgroups of $C(U)$. Then*

$$\mathbf{F}[G_1 G_2] = \mathbf{F}[G_1] \cap \mathbf{F}[G_2]$$

where $G_1 G_2$ denotes the smallest subgroup containing both G_1 and G_2.

(*c*) *Let $C_{\mathrm{fin}}(U) \subset C(U)$ denote the set of elements of $C(U)$ of finite order. If $\mathrm{EC}(U)$ is finite then $C(U)$ is a group and $C_{\mathrm{fin}} := C_{\mathrm{fin}}(U)$ is a finite subgroup. Consequently*

$$|G_1 G_2| \le |C_{\mathrm{fin}}| < \infty$$

for any finite subgroups G_1, G_2 of $C(U)$.

PROOF: Choose some set $B \in \mathcal{Y}$ satisfying $0 < \nu(B) < 1/|G|$. Then $\bigcup_{S \in G} S^{-1}(B)$ is a non-trivial G-invariant set. This establishes *(a)*.

Part *(b)* is a tautology. Part *(c)* is a consequence of theorem 4.7 and diagram 6.7. Indeed, C_{fin} is the kernel "$\ker(\varphi)$" of diagram 6.7 and so has cardinality no more than $|\mathrm{EC}(U)|$. ◊

Now let us return to our finite joining-rank T and its maximal-joining $\breve{\mu}$. By the preceding lemma we know that $C_{\mathrm{fin}} \subset C(\breve{T})$, the subset of elements of finite order, is a finite subgroup.

LEMMA 7.11. *Suppose $\mathcal{F}$ is a component-factor of $(\breve{T}\colon\breve{X},\breve{\mathcal{X}},\breve{\mu})$. Let $G := \Gamma[\mathcal{F}]$ denote the* ***stabilizer*** *of $\mathcal{F}$*

$$\Gamma[\mathcal{F}] := \{S \in C(\breve{T}) \mid \forall A \in \mathcal{F},\, S^{-1}(A) = A\}$$

which is a subgroup of C_{fin}. Then $\mathbf{F}[G] = \mathcal{F}$.

PROOF: Any S stablizing a component-factor (by definition non-trivial) has finite order. For letting k denote the order of permutation π of (7.9c), S^k equals $\breve{T}^n$ for some n. Since T is totally ergodic this $\breve{T}^n$ can not fix a component-factor, unless $n = 0$. Thus $\Gamma[\mathcal{F}] \subset C_{\text{fin}}$.

To show equality of $\mathbf{F}[G]$ with $\mathcal{F}$, we need but show $\mathbf{F}[G] \subset \mathcal{F}$. Let $\rho := \breve{\mu}_{\langle 1\rangle} \times_{\mathcal{F}} \breve{\mu}_{\langle 2\rangle}$ be the relative independent joining over $\mathcal{F}$ and let ξ be an ergodic component of ρ. By (7.9) ξ is a graph joining

$$\xi = [\![I \times S]\!]\Delta_{\breve{\mu}} \qquad \text{for some } S \in C(\breve{T}).$$

By definition $\rho\lfloor_{\mathcal{F}_{\langle 1\rangle}\times\mathcal{F}_{\langle 2\rangle}} = \Delta_{\breve{\mu}}|_{\mathcal{F}\times\mathcal{F}}$, which is isomorphic to $\breve{\mu}|_{\mathcal{F}}$. Since this latter is ergodic, $\xi\lfloor_{\mathcal{F}_{\langle 1\rangle}\times\mathcal{F}_{\langle 2\rangle}}$ equals $\rho\lfloor_{\mathcal{F}_{\langle 1\rangle}\times\mathcal{F}_{\langle 2\rangle}}$. Thus

$$([\![I \times S]\!]\Delta_{\breve{\mu}})|_{\mathcal{F}\times\mathcal{F}} = \Delta_{\breve{\mu}}|_{\mathcal{F}\times\mathcal{F}}$$

and so S fixes every member of $\mathcal{F}$. In other words, $S \in G$.

Any invariant measure is an integral over its ergodic components—so there is some probability measure, call it η, on G such that

$$\rho(A \times B) = \int_G [\![I \times S]\!]\Delta_{\breve{\mu}}\,(A \times B)\; d\eta(S), \qquad A, B \in \breve{\mathcal{X}}.$$

[Since G is a finite set this integral is just a finite sum.] In particular, for $B \in \mathbf{F}[G]$

$$\rho(B \times B) = \int_G \breve{\mu}(B \cap S^{-1}(B))\, d\eta(S) = \int_G \breve{\mu}(B \cap B)\, d\eta(S) = \breve{\mu}(B).$$

Observation (I.3) applies and indicates that $B \in \mathcal{F}$. ◊

The prime factor is unique. We now have all the tools needed to finish up the proof of the *Canonical Structure Theorem.*

UNIQUENESS PROPOSITION, 7.12. *For transformation $(\breve{T}\colon\breve{X},\breve{\mathcal{X}},\breve{\mu})$ suppose $\mathcal{P}_i \subset \breve{\mathcal{X}}$ is a component-factor which is prime, for $i = 1, 2$. Then $\mathcal{P}_1 = \mathcal{P}_2$.*

PROOF: Let $G_i := \Gamma[\mathcal{P}_i]$, a finite group. Thus G_1G_2 is finite and therefore

$$\mathbf{F}[G_1G_2] = \mathbf{F}[G_1] \cap \mathbf{F}[G_2] = \mathcal{P}_1 \cap \mathcal{P}_2$$

is a non-trivial factor of $\breve{T}$, by the Commutant Lemma above. Thus $\mathcal{P}_1 = \mathcal{P}_2$. ◊

We know that $(T\colon X, \mu)$ has at least one prime factor; suppose it had two, $\mathcal{P}$ and $\mathcal{P}'$. We can view these as factor fields of $\mathcal{X}_{\langle 1\rangle}$ in the maximal-joining $\breve{\mu}$. Hence $\mathcal{P} = \mathcal{P}'$. Thus T has a *unique* prime factor field.

Lastly we need to show that the S of the Rudimentary Structure Theorem has minimal self-joinings. Renaming S to T, we have that T is prime with trivial commutant and wish to show that $R := \mathrm{jr}(T)$ equals 1. But in the maximal-joining $\breve{\mu}$ for T, the component-factors $\mathcal{X}_{\langle 1\rangle}, \ldots, \mathcal{X}_{\langle R\rangle}$ are each prime; hence

$$\mathcal{X}_{\langle 1\rangle} = \mathcal{X}_{\langle 2\rangle} = \cdots = \mathcal{X}_{\langle R\rangle}.$$

So each pair of fields must be joined by a graph-joining of some element of $C(T)$. But $C(T) = \{T^n\}_{n\in\mathbb{Z}}$ and so only the off-diagonal joinings are available. Since they are forbidden in the maximal-joining, R must be 1. ◊

§8 Simplicity and Glasner's example

The preceding proof takes pains to never assume that the maximal-joining is weak-mixing. As we shall see below, it need not be.

Using a circle extension of a circle extension, Eli Glasner constructed in Proposition 1.7 of [**G**] an ergodic non-weak-mixing self-joining of a weak-mixing transformation. We recapitulate the example here, using Chacón's transformation for the base and the 2-point group $\mathbb{Z}_2 = \{\mathbf{-1}, \mathbf{1}\}$ (written multiplicatively) in place of the circle group. The cocycle will be described explicitly.

Equip $\mathbb{Z}_2$ with Haar measure. Let $(U\colon Y, \nu)$ be Chacón's transformation and let $X' := Y \times \mathbb{Z}_2$ and $X := Y \times \mathbb{Z}_2 \times \mathbb{Z}_2$ with μ' and μ denoting the corresponding product measures. For a cocycle $c\colon Y \to \mathbb{Z}_2$ define $T'\colon X' \to X'$ and T on X by

$$T'(y,a) := Uy, ac(y)$$
$$T\big(y,a,b\big) := Uy, ac(y), ba$$

for $a, b \in \mathbb{Z}_2$. Thus T is a group extension of T' which is, in turn, a group extension of U. Suppose that a cocycle can be found so that this T is weak-mixing. Since $\mathrm{jr}(U) = 1$ by [**J,R,S**], theorem I.8 insures that $\mathrm{jr}(T)$ equals four.

Define the "flip" $F \in C(T)$ by

$$F\big(y,a,b\big) := y, a, -b.$$

Define a $\rho \in \mathbb{J}(T,T)$ as the average of two (non-invariant) graph joinings:

$$\rho = \tfrac{1}{2}\big[\!\big[I \times \Phi^{+}\big]\!\big]\Delta_{\mu} + \tfrac{1}{2}\big[\!\big[I \times \Phi^{-}\big]\!\big]\Delta_{\mu} \tag{8.1}$$

where $\Phi^{\pm}(y,a,b) := (y, -a, \pm b)$. Both Φ^{+} and Φ^{-} preserve μ and their commutation relation with T is $\Phi^{\pm} \circ T = T \circ \Phi^{\mp}$. Hence the square, $\big((T{\times}T)^2\colon \rho\big)$, is not ergodic; its two ergodic components are the righthand side of (8.1). On the other hand, $(T{\times}T\colon \rho)$ *is* ergodic since any ergodic component would have to be invariant under $(T{\times}T)^2$; yet the action of $T{\times}T$ exchanges the two graph joinings on the righthand side of (8.1).

The maximal-joining. The upshot is that transformation $(T{\times}T\colon \rho)$ has a factor which is rotation on a two-point space. Since $\breve{T}$ will be an extension of this transformation it too has such a factor. One may check that the maximal-joining is

$$\breve{\mu}(A_1{\times}A_2{\times}A_3{\times}A_4) := \rho\big((A_1 \cap F^{-1}A_2) \times (A_3 \cap F^{-1}A_4)\big).$$

All the non-trivial 2-fold self-joinings of T can be observed among the six 2-fold marginals of $\breve{\mu}$:

$$\begin{array}{ccc} \mathcal{X}_1 & \xrightarrow{[\![I\times F]\!]\Delta_\mu} & \mathcal{X}_2 \\ {\scriptstyle [\![I\times F]\!]\rho}\big\downarrow & & \big\uparrow{\scriptstyle [\![I\times F]\!]\rho} \\ \mathcal{X}_4 & \xleftarrow[{[\![I\times F]\!]\Delta_\mu}]{} & \mathcal{X}_3 \end{array} \qquad \text{Diagonal marginals: } \breve{\mu}|_{1,3} \cong \rho \cong \breve{\mu}|_{2,4}\ .$$

For an arrow, above, labelled with a joining "ν", the "arrow end" points to the field which is the second coordinate of ν and the "feather end" points to the first coordinate of ν. Each diagonal of the diagram is coupled by ρ, which is a symmetric joining. (It is invariant under exchange of its first and second coordinates.)

This example demonstrates the necessity that the factor field $\mathcal{F}$ of Lemma 7.5 be a component-factor. The above two-point rotation factor is independent of each subfield $\mathcal{X}_i$.

Making the cocycle. Chacón's transformation is described below (5.6′). Let H_n denote the n-block and h_n its length; h_0 is zero. Henceforth, "n" ranges only over *even* values. Thus each h_n is even, since $h_{n+2} = 9h_n + 4$. Referring to figure (5.7),

$$H_{n+2} \;=\; \big(H_n\;\;H_n\,1_n\,H_n\big)\,\big(H_n\;\;H_n\,2_n\,H_n\big)\,3_n\,\big(H_n\;\;H_n\,4_n\,H_n\big) \tag{$*$}$$

where we have numbered the four spacers $1_n, \ldots, 4_n$. Cocycle $c(\cdot)$ will be constant on levels of each n-stack. Having inductively defined $c(\cdot)$ on H_n, let

$$c(1_n) = c(3_n) := +\mathbf{1} \qquad \text{and} \qquad c(2_n) = c(4_n) := -\mathbf{1}. \tag{8.2i}$$

What follows is a sketch of a standard type of argument that an extension is weak-mixing.

Traversing an n-block has no net effect on the bit-pair. Let $\mathrm{Bits}(y,a,b)$ denote the ordered pair (a,b). For $i \in [0 \mathinner{..} h_n]$, let "$T_n^i(a,b)$" abbreviate $\mathrm{Bits}\big(T^i(y,a,b)\big)$, where y is a point in the base of H_n; this is independent of y since the cocycle is constant on levels.

Induction on n establishes

$$\prod_{i=0}^{h_n-1} c\big(\text{Level } i \text{ of } H_n\big) = \mathbf{1}.$$

Equivalently, $T_n^{h_n}(a,?) = (a,?)$. A second induction on n now shows the following.

$$\forall a,b \in \mathbb{Z}_2 : \qquad T_n^{h_n}(a,b) = (a,b) \tag{8.2ii}$$

Ergodicity. T has four "types" of n-block, one for each ordered pair $a,b \in \mathbb{Z}_2$. The n-block $H_n^{a,b}$ is the sequence of bit-pairs.

$$T_n^0(a,b)\;\;T_n^1(a,b)\;\;\ldots\;\;T_n^{h_n-1}(a,b).$$

Having chosen type (a,b) for the $(n+2)$-block of $(*)$, equations (8.2) determine the types of its nine constituent n-blocks. They turn out to be, in addition to (a,b), the three types below on the right.

$$(a,b) \quad\longrightarrow\quad (-a,b),\;(a,ab),\;(-a,-ab). \tag{8.3}$$

To prove T ergodic it suffices to show, for some constant K, that each type of n-block contains all four types of $(n-2K)$-block. So viewing $(\mathbf{1},\mathbf{1})$, $(-\mathbf{1},\mathbf{1})$, $(-\mathbf{1},-\mathbf{1})$ and $(\mathbf{1},-\mathbf{1})$ as the vertices of a digraph and putting in directed edges according to (8.3), we need but check that the resulting digraph is connected; following the arrows, one can go from any vertex to any other. Indeed, only arrows $(-a,b)$ and (a,ab) are needed to show that $K=3$ works.

Weak-mixing. It suffices to rule out any non-one eigenvalue. But notice that the leftmost three n-blocks of $H_{n+2}^{\mathbf{1},\mathbf{1}}$ form the pattern

$$H_n^{\mathbf{1},\mathbf{1}}\;\;H_n^{\mathbf{1},\mathbf{1}}\mid H_n^{\mathbf{1},\mathbf{1}}$$

where "|" represents a spacer. The standard Chacón argument now applies and shows that $\lambda = 1$ is the only eigenvalue. ◊

Final remarks. For a T of joining-rank R, its maximal extension $\breve{T}$ can not have finite joining-rank if it is not weak-mixing. Conversely if it is, part (b) of the Maximal-joining Lemma says that

Any $\xi \in \mathbb{J}_{\mathrm{Erg}}(\breve{T}, \breve{T})$ *which is not product measure, is a graph joining.*

Such a transformation $\breve{T}$ is said to be (2-fold) ***simple***. Evidently, a weak-mixing simple map has joining-rank equal to the size of its essential commutant. So if the maximal-joining is weak-mixing then

$$\mathrm{jr}(\breve{T}) = \#\,\mathrm{EC}(\breve{T}) \leq \big[\mathrm{jr}(T)\big]!$$

§A Appendix

Disjointness. An ergodic map is "disjoint" from any identity map in the following sense. If $(T\colon X,\mu)$ is ergodic and I is the identity map on (Y,ν) then the only joining $\xi \in \mathbb{J}(T,I)$ is product measure $\mu\times\nu$.

Letting $\mathbb{E}$ denote conditional expectation with respect to ξ, this will follow from showing, for each bounded function $f\colon Y \to \mathbb{R}$, that $\mathbb{E}_{\mathcal{X}}(1_X \times f)$ is a constant function on X; this will imply that the fiber measures of ξ over $\mathcal{X}$ are all equal. But $1_X\times f$ is $T\times I$ invariant and so is ξ. Thus $\mathbb{E}_{\mathcal{X}}(1_X\times f)$ is T-invariant, hence constant.

Algebraic Extensions. Suppose $(U\colon Y,\nu)$ is an ergodic transformation and $(\mathbb{G},\mathcal{G},\eta)$ represents a compact topological group, its Borel field, and Haar measure. Haar measure is both left and right invariant since $\mathbb{G}$ is compact. A ***group extension*** is a transformation T defined on the space $(Y \times \mathbb{G}, \mathcal{Y} \times \mathcal{G}, \nu \times \eta)$ determined by a measurable map $c\colon Y \to \mathbb{G}$ as follows.

$$y, h \xrightarrow{T} U(y), hc(y) \qquad y \in Y, h \in \mathbb{G} \tag{A.1}$$

where juxtaposition "$hc(y)$" indicates multiplication in the group. Evidently there is a subgroup $\widehat{\mathbb{G}}$ of $C(T)$ isomorphic to $\mathbb{G}$: For $g \in \mathbb{G}$ define $\widehat{g}(y,h) := (y, gh)$.

Isometric Extensions. In an isometric extension, the fiber space is a compact metric space M on which acts a transitive group $\mathbb{G}$ of isometries. Then M can be identified with a quotient of $\mathbb{G}$. Here, we will take as our definition that the fiber space is a quotient.

Given a compact metrizable topological group $(\mathbb{G},\mathcal{G},\eta)$ and closed subgroup $\Gamma \subset \mathbb{G}$, the quotient space $M := \Gamma\backslash\mathbb{G}$ of right cosets inherits the metric with distance from Γg_1 to Γg_2 being the infimum of $\mathrm{dist}(x_1,x_2)$ over all $x_i \in \Gamma g_i$. This makes the projection map, $\pi\colon \mathbb{G} \to M$ sending $h \mapsto \Gamma h$, continuous. Letting $\mathcal{M}$ denote its Borel field, $(M,\mathcal{M},\overline{\eta})$ is a probability space, where $\overline{\eta} := \eta \circ \pi^{-1}$. Each element $g \in \mathbb{G}$ acts (from the right) on M by $\Gamma h \xrightarrow{g} \Gamma hg$. This action preserves $\overline{\eta}$.

An ***isometric extension,*** $\overline{T}$, of $(U\colon Y,\nu)$ is specified by a measurable cocycle $c\colon Y \to \mathbb{G}$. The mapping

$$y, m \xrightarrow{\overline{T}} U(y), mc(y) \qquad y \in Y, m \in M \tag{A.2}$$

defines $\left(\overline{T}\colon Y\times M, \nu\times\overline{\eta}\right)$. This extension is a factor of the group extension T defined in (A.1) via the homomorphism $y, h \longmapsto y, \Gamma h$. Even if $\overline{T}$ is ergodic the larger extension T need not be.

Theorem A.3. *Any ergodic isometric extension of U is a product lift of U.*

Remark. The referee informs us that this theorem is essentially equivalent to a result of [**F2**], and that [**G,W**] contains variations on the "product lift" notion. The result also appears in [**L**].

Proof: Fix some weak-mixing $(S\colon W,\lambda)$.

Let us first assume that the extension is a group extension as in (A.1). Consider any probability measure φ on $(\mathbb{G},\mathcal{G})$. For arbitrary $B \in \mathcal{G}$ and $g \in \mathbb{G}$

$$\varphi(g^{-1}B) = \int 1_{g^{-1}B}(h)\, d\varphi(h) = \int 1_{Bh^{-1}}(g)\, d\varphi(h)$$

where, as will be the case below, integrals with no subscript are taken over the group $\mathbb{G}$. The equality above and Fubini's theorem produce

$$\begin{aligned}\int_{\mathbb{G}} \varphi(g^{-1}B)\,d\eta(g) &= \iint 1_{Bh^{-1}}(g)\,d\eta(g)\;d\varphi(h)\\ &= \int \eta(Bh^{-1})\;d\varphi(h) \overset{\text{Haar}}{=\!=\!=} \int \eta(B)\,d\varphi(h) = \eta(B).\end{aligned} \tag{$*$}$$

Now fix a joining $\xi \in \mathbb{J}_{\mathrm{Erg}}(T,S)$ such that $\xi\lfloor\mathcal{Y}\vee\mathcal{W} = \nu\times\lambda$. Our goal is to show that ξ equals $(\nu\times\eta)\times\lambda$.

Each $\widehat{g}$ is in $C(T)$ and the identity map I is in $C(S)$. Thus

$$\xi_g := [\![\widehat{g}\times I]\!]\xi$$

is isomorphic to ξ and consequently is an ergodic joining of T with S. For sets $Y'\in\mathcal{Y}, B\in\mathcal{G}$ and $W'\in\mathcal{W}$ its value is

$$\xi_g\big((Y'\times B)\times W'\big) = \xi\big((Y'\times g^{-1}B)\times W'\big) = \int_{Y'\times W'} \varphi_{y,w}(g^{-1}B)\;d\,\nu{\times}\lambda\,(y,w)$$

where $\{\varphi_{y,w}\}$ denotes the fiber measure decomposition of ξ over its $\mathcal{Y}\times\mathcal{W}$ factor. Average the $\{\xi_g\}_g$ to form a joining $\alpha(\cdot) := \int_{\mathbb{G}}\xi_g(\cdot)\,d\eta(g)$, which is not necessarily ergodic. Thus

$$\begin{aligned}\alpha\big((Y'\times B)\times W'\big) &= \int_{\mathbb{G}}\int_{Y'\times W'} \varphi_{y,w}(g^{-1}B)\;d\,\nu{\times}\lambda\,(y,w)\;d\eta(g)\\ &\overset{\text{by }(*)}{=\!=\!=} \int_{Y'\times W'} \eta(B)\;d\,\nu{\times}\lambda\,(y,w)\end{aligned}$$

which equals $\nu(Y')\eta(B)\lambda(W')$. In other words, α is product measure $(\nu\times\eta)\times\lambda$.

But product measure is ergodic, since T is ergodic and S is weak-mixing. Hence $\alpha = \xi_g$ for η-a.e. g. For such a g,

$$\xi = [\![\widehat{g^{-1}}\times I]\!]\xi_g = [\![\widehat{g^{-1}}\times I]\!]\alpha = \alpha$$

since product measure is unaffected by translation in the fiber direction.

Full Generality. Suppose now that the given ergodic isometric extension is as in (A.2) and let $\overline{X} := Y{\times}M$ and $\overline{\mu} := \nu{\times}\overline{\eta}$ denote its space and measure. It is a factor of group extension $(T\colon X,\mu)$ of (A.1) where $X := Y{\times}\mathbb{G}$ and $\mu := \nu{\times}\eta$. Given a joining $\overline{\xi}\in\mathbb{J}(\overline{T},S)$ for which

$$\mathcal{Y}\perp^{\overline{\xi}}\mathcal{W},$$

define the relative independent joining $\xi\in\mathbb{J}(T,S)$ of

$$(T\colon\mu)\longrightarrow(\overline{T}\colon\overline{\mu})\longleftarrow(\overline{T}{\times}S\colon\overline{\xi})$$

over their common factor, as in (I.6). Thus

$$\xi|_{\mathcal{Y}\vee\mathcal{W}} = \overline{\xi}|_{\mathcal{Y}\vee\mathcal{W}} = \nu\times\lambda.$$

This ξ need not be ergodic.

Mimicking the argument for a group extension, define $\xi_g := [\![\widehat{g} \times I]\!]\xi \in \mathbb{J}(T, S)$ and $\alpha(\cdot) := \int_{\mathbb{G}} \xi_g(\cdot)\, d\eta(g)$ to conclude that $\mathcal{X} \perp^\alpha \mathcal{W}$. Thus

$$\begin{aligned} \overline{\mu} \times \lambda &= \alpha|_{\overline{\mathcal{X}} \vee \mathcal{W}} \\ &= \xi_g|_{\overline{\mathcal{X}} \vee \mathcal{W}} \qquad \text{for } \eta\text{-a.e. } g \in \mathbb{G} \end{aligned}$$

where the second inequality –since $\overline{\mu} \times \lambda$ is ergodic– follows from the uniqueness in the ergodic decomposition theorem.

But $\alpha = [\![\widehat{h} \times I]\!]\alpha$ for any $h \in \mathbb{G}$. So for a g as above,

$$\alpha\big|_{\overline{\mathcal{X}} \vee \mathcal{W}} = [\![\widehat{g^{-1}} \times I]\!]\xi_g\big|_{\overline{\mathcal{X}} \vee \mathcal{W}} = \xi\big|_{\overline{\mathcal{X}} \vee \mathcal{W}}$$

which equals $\overline{\xi}$. Thus $\overline{\xi}$ equals $\overline{\mu} \times \lambda$, as desired. ◊

References

[F1] H. Furstenberg, "Recurrence in Ergodic Theory and Combinatorial Number Theory," Princeton University Press, 1981.

[F2] H. Furstenberg, *Strict ergodicity and transformations of the torus*, Amer. J. Math. **83** (1961), 573–601.

[G] S. Glasner, *Quasi-factors in ergodic theory*, Israel J. Math. **45** (1983), 198–208.

[G,W] S. Glasner, B. Weiss, *Processes disjoint from weak mixing*, Trans. Amer. Math. Soc. **316** (1989), 689–703.

[J,R] A. del Junco, D.J. Rudolph, *On ergodic actions whose self-joinings are graphs*, Ergodic Theory and Dynamical Systems **7** (1987), 531–557.

[J,R,S] A. del Junco, A.M. Rahe, L. Swanson, *Chacón's automorphism has minimal self-joinings*, J. Analyse Math. **37** (1980), 276–284.

[L] E. Lesigne, *Un théoreme de disjonction de systèmes dynamiques et une généralisation du théoreme ergodique de Wiener-Wintner*, Ergodic Theory and Dynamical Systems **10** (1990), 513–521.

[R] D.J. Rudolph, *An example of a measure preserving map with minimal self-joinings, and applications*, J. Analyse Math. **35** (1979), 98–122.

[Z1] R. Zimmer, *Extensions of ergodic group actions*, Illinois J. Math. **20** (1976), 373–409.

[Z2] R. Zimmer, *Ergodic actions with generalized discrete spectrum*, Illinois J. Math. **20** (1976), 555–588.

[1] J.L. King, *Joining-rank and the structure of finite rank mixing transformations*, J. Analyse Math. **51** (1988), 182–227.

[2] J.L. King, *The commutant is the weak-closure of the powers, for rank-1 transformations*, Ergodic Theory and Dynamical Systems **6** (1986), 363–384.

[3] N. Friedman, P. Gabriel, J.L. King, *An invariant for rank-1 rigid transformations*, Ergodic Theory and Dynamical Systems **8** (1988), 53-72.