Problems assigned in MAP 6472

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1. Show that $\forall s > 0$, $\int_{-\infty}^{\infty} e^{-\frac{x^2}{s}} = \sqrt{s\pi}$.
2. Let $A := \{3, 30, 31, 32, 33, \ldots, 39, 300, 301, \ldots, 399, 3000, 3001, \ldots\}$. What is $\overline{\mathrm{Den}}(A)$ and $\underline{\mathrm{Den}}(A)$? Does $\mathrm{Den}(A)$ exist?
3. Let $\{T_k\}$ be a increasing sequence of posints such that $\frac{T_{k+1}-T_k}{T_k} \to 0$ as $k \to \infty$. Show that for any set of posints S, $\mathrm{Den}(S)$ along $T_k = \mathrm{Den}(S)$.
4. Let $\mathcal{V} := \{\omega \in \Omega : \exists \;\text{ an }\; N \;\text{ such that }\; \forall\; k \geq N \;\; \omega_k = 1\}$. Find an infinite sequence of words $\{W_k\}$ such that $\mathcal{V} \subset \cup_{k=1}^{\infty}[W_k]$ and $\sum_{k=1}^{\infty} P([W_k]) < \frac{1}{2003}$.
5. Let $\{T_k\}$ be a sequence of posints. Let $B := \{\omega \in \Omega : \exists \;\text{ an }\; N \;\text{ such that }\; \forall \;\; k \geq N \;\text{ we have }\; \omega_{T_k} = 1\}$. Show that $\forall \;\; \epsilon > 0 \;\; \exists$ a sequence of words $\{W_n\}$ such that $B \subset \cup_{n=1}^{\infty}[W_n]$ and $\sum_{n=1}^{\infty} \Pr([W_n]) < \epsilon$.
6. Problem 14 on page 34 of text.
7. Fix functions $f_m : \Omega \to \mathbf{R}$. Let $C_m(\epsilon) := \{f_m > \epsilon\}$ which is notation for $\{\omega \in \Omega : f_m(\omega) > \epsilon\}$. Let $S(\epsilon) := \{[\limsup_{m\to\infty} f_m] > \epsilon\}$. Prove that
$$S(\epsilon) = \bigcup_{n=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{m=k}^{\infty} C_m(\epsilon + \frac{1}{n}).$$
8. Given a set Ω, index sets B, A such that $S_{\beta,\alpha} \subset \Omega$ for all $\alpha \in A, \beta \in B$. Prove that
$$\bigcap_{\beta\in B} \bigcup_{\alpha\in A} S_{\beta,\alpha} = \bigcup_{f\in A^B} \bigcap_{\beta\in B} S_{\beta,f(\beta)}$$
where $A^B := \{f : f \;$ is a function with domain B and range $A\}$.
9. Let $\Omega := \mathcal{A}_1 \times \mathcal{A}_2 \times \ldots$ be a countable product of finite alphabets. You can assume each $\mathcal{A}_i := \{H, T\}$. Then Ω will be the coin-flip space. Suppose that $\{W_i\}$ is an infinite sequence of words, no one of which is a prefix of another (so that their cylinder sets are disjoint). Show that the union of their cylinder sets
$$\bigcup_{i=1}^{\infty}[W_i] \neq \Omega.$$
In other words if you write Ω as a countable union of disjoint cylinder sets, then secretly the union is actually finite.

10. (a) For a real number x, let $G(x) := \exp -x^2$. Compute the points of inflection of $G(x)$.

 (b) Where are the points of inflection of $G_S(x) = \exp \frac{-x^2}{S}$ where S is a positive real number?

 (c) Let $N := 500$. For $k = 0, 1, 2, \ldots, 500$, let $P(k)$ be the probability of getting exactly k Heads when independently flipping 500 coins. The graph of $k \mapsto P(k)$ is approximately a Gaussian distribution. What probabilistic interpretation can you give to the "inflection points" of $k \mapsto P(k)$?

13. **A Philosophical Question** For a game G, let $\mathcal{D}$ be the set of deterministic strategies and $\mathcal{P}$ be the set of probabilistic strategies. Since $\mathcal{D} \subset \mathcal{P}$, we have

$$\sup_{D \in \mathcal{D}} E[\text{Strategy } D] \leq \sup_{S \in \mathcal{P}} E[\text{Strategy } S],$$

where $E[\cdot]$ denotes the expected winnings.

 (a) Discuss the *non-strictness* of the above inequality. In other words, is it necessarily the case that the above inequality is not strict?

 (b) Does there exist a strategy $S \in \mathcal{P}$ such that $E[S] = \sup_{D \in \mathcal{D}} E[D]$?

14. Fix a real number $r > 0$ and let A_r be the set of $x \in \Omega := [0, 1]$ so that there are infinitely many pairs of integers (N, Q) with Q positive and coprime with N, such that

$$\left| x - \frac{N}{Q} \right| \leq \frac{1}{Q^r}. \tag{1}$$

For what values of r can you use the Borel-Cantelli lemma to show that $\Pr(A_r) = 0$?

Hint: Let B_Q be the set of x for which there exists N fulfilling (1).

15. Fix non-negative reals p, q such that $p + q = 1$ and $p > \frac{1}{2}$. Flip a p, q coin independently. It outputs "+1" with probability p and "-1" with probability q. Let S_n be the sum of the first n coin-flips. (So this is a "random walk on the integers".) Show that the event of "returning to the origin infintely-often" is null i.e.

$$\Pr(\{S_n = 0 \text{ for } \infty\text{-ly many } n\}) = 0.$$

Hint: For $n = 2k$, let B_k be the event that $S_{2k} = 0$ i.e you have returned to the origin at time $2k$. Recall that we used Stirling's formula to estimate the binomial coefficient $\binom{2k}{k}$. Use this estimate, together with the fact that $p \neq \frac{1}{2}$ to show that $\Pr(B_k)$ decreasing exponentially to zero. Thus $k \mapsto \Pr(B_k)$ is summable. Now apply Borel-Cantelli Lemma.

16. Find a random process $\vec{Z}$ on $(\Omega, \mathcal{F}, P)$and a random variable W such that

 (a) For all posints n, $Z_n \geq 0$ and $E[Z_n] \leq 17$.

 (b) $Z_n \overset{\text{a.s.}}{\to} W$.

 (c) There is no a.s. convergent subsequence $\vec{X}$, where $X_n := E[Z_n|\mathcal{G}]$ and $\mathcal{G} := \{B \times \Omega : B \in \mathcal{F}\}$.

17. Suppose X, Y are random variables such that $E[X^2] < \infty$ and $E[Y^2] < \infty$, then prove that

$$E[X \cdot Y|\mathcal{G}]^2 \leq E[X^2|\mathcal{G}] \cdot E[Y^2|\mathcal{G}]. \tag{2}$$