RREF is unique

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When is a matrix in RREF? In a $\mathsf{B} = [b_{i,j}]_{i,j}$ matrix, row-i being NotAZ means it is Not-All-Zero; let Col(i) be the column-index of its *leftmost* non-zero entry; thus $b_{i,\,\mathrm{Col}(i)}$ is the leftmost non-zero entry in row-i.

Our B *is in RREF* (***reduced row-echelon-form***) *if*

i: The NotAZ rows are above the All-Zero rows.

ii: With P denoting the number of NotAZ rows, we have
$$\mathrm{Col}(1) < \mathrm{Col}(2) < \ldots < \mathrm{Col}(P)\,.$$

iii: For $i = 1, \ldots, P$, the *only* non-zero entry in column Col(i) is at position (i, Col(i)). Moreover $b_{i,\,\mathrm{Col}(i)} = 1$.

For $i = 1, 2, \ldots, P$, we call row-i a ***pivot row***, column Col(i) a ***pivot column***, and position (i, Col(i)) a ***pivot position***.

The *RREF* of a matrix is unique. However, removing "reduced" gives **row-echelon-form**, and different textbooks have slightly varying definitions of REF. While *RREF* is unique, REF is *not* unique. Nonetheless, useful properties can be read-off from an REF of a matrix. □

See next page. . .

1: RREF Uniqueness Thm. *Consider two matrices* A *and* B *in RREF, having the same dimensions and over the same field* F. *If* $\mathsf{A} \overset{r}{\sim} \mathsf{B}$ [*row-equiv.*] *then* A = B. ◊

Key idea. Row-ops (elementary row-operations) do not change linear relations among columns. □

Proof. FTSO *Contradiction*, suppose $\mathsf{A} \neq \mathsf{B}$.

Let $\boldsymbol{\alpha}(j)$ denote the j^{th} column of A; ditto $\boldsymbol{\beta}(j)$ for B. Take index K *smallest* such that $\boldsymbol{\alpha}(K) \neq \boldsymbol{\beta}(K)$.

Let P denote the number of pivot columns that A, hence B (Exer: Why?), has to the *left* of column-K. Write A's pivot-positions as

$$(1, c_1), (2, c_2), \ldots, (P, c_P),$$

where $c_1 < \cdots < c_P$ are the column indices.

Case: Column $\boldsymbol{\alpha}(K)$ is a non-pivot column

Then $\boldsymbol{\alpha}(K)$ has form $\begin{bmatrix} x_1 \\ \vdots \\ x_P \\ 0 \\ \vdots \\ 0 \end{bmatrix}$. Thus column $\boldsymbol{\alpha}(K)$ is a linear-combination of the pivot-columns to its left, namely

$$\boldsymbol{\alpha}(K) \quad = \quad \sum_{\mathsf{i}=1}^{P} x_{\mathsf{i}} \cdot \boldsymbol{\alpha}(c_{\mathsf{i}}).$$

Recall row-equivalence *preserves* linear relations among columns, [ie., LNul(A) = LNul(B)] hence

$$\boldsymbol{\beta}(K) \quad = \quad \sum_{\mathsf{i}=1}^{P} x_{\mathsf{i}} \cdot \boldsymbol{\beta}(c_{\mathsf{i}}).$$

But to the left of column-K, matrices A and B *agree*. For each index $j < K$, consequently, $\boldsymbol{\alpha}(j) = \boldsymbol{\beta}(j)$. In particular, each $\boldsymbol{\alpha}(c_{\mathsf{i}}) = \boldsymbol{\beta}(c_{\mathsf{i}})$. Thus

$$\boldsymbol{\beta}(K) \quad = \quad \sum_{\mathsf{i}=1}^{P} x_{\mathsf{i}} \cdot \boldsymbol{\alpha}(c_{\mathsf{i}}) \quad \overset{\text{note}}{=\!=} \quad \boldsymbol{\alpha}(K),$$

contradicting that $\boldsymbol{\alpha}(K)$ is unequal to $\boldsymbol{\beta}(K)$.

Case: Column $\boldsymbol{\alpha}(K)$ *is* a pivot-column Since the preceding argument also applies to matrix B, column $\boldsymbol{\beta}(K)$ must itself be a pivot-column (of B, of course).

As $\boldsymbol{\alpha}(K)$ is a pivot-column with P many pivots to its left, necessarily our $\boldsymbol{\alpha}(K)$ equals the transpose of

$$*: \qquad \big[\overbrace{0 \ \ldots \ 0}^{P \text{ many}} \ 1 \ 0 \ \ldots \ 0\big].$$

But column $\boldsymbol{\beta}(K)$ is also a pivot column, and it also has P many pivots to *its* left. So $\boldsymbol{\beta}(K)$ equals $(*)$, hence equals $\boldsymbol{\alpha}(K)$; again a contradiction. ♦

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