Rank + Nullity theorem

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Entrance. Here, $\mathbf{V}, \mathbf{H}$ are finite-dimensional VSes over the same field, and $\mathsf{T}: \mathbf{V} \to \mathbf{H}$ is linear. Recall

$$\begin{aligned} \text{Nullity}(\mathsf{T}) &:= \text{Dim}(\text{Nul}(\mathsf{T})), \quad \text{and} \\ \text{Rank}(\mathsf{T}) &:= \text{Dim}(\text{Range}(\mathsf{T})). \end{aligned}$$

1: Rank-Nullity Thm. *For* $\mathsf{T}: \mathbf{V} \to \mathbf{H}$ *as above,*

$$\begin{aligned} \text{Rank}(\mathsf{T}) + \text{Nullity}(\mathsf{T}) &= \text{Dim}(\text{Dom}(\mathsf{T})) \\ &\stackrel{here}{=} \text{Dim}(\mathbf{V}). \end{aligned}$$ ◊

Proof. Choose a basis $\mathcal{Z} = \{\mathbf{z}_1, \ldots, \mathbf{z}_N\}$ for $\text{Ker}(\mathsf{T})$. [Mnemonic: $\mathcal{Z}$ is for "zero".] Pick a basis

$$\mathcal{H} = \{\mathbf{h}_1, \ldots, \mathbf{h}_R\}$$

for $\text{Range}(\mathsf{T})$. Finally, for each index $r \in [1..R]$ pick a vector $\widehat{\mathbf{h}}_r \in \mathsf{T}^{-1}(\mathbf{h}_r)$; possible, since each $\mathbf{h}_r$ is in $\text{Range}(\mathsf{T})$.

Our goal is to show that list

$$\mathcal{B} := (\mathbf{z}_1, \ldots, \mathbf{z}_N, \widehat{\mathbf{h}}_1, \ldots, \widehat{\mathbf{h}}_R)$$

is a $\mathbf{V}$-basis.

Proof: $\mathcal{B}$ is LI. Consider a linear-combination

$$*: \quad \Big[\sum_{k=1}^N \beta_k \mathbf{z}_k\Big] + \Big[\sum_{r=1}^R \alpha_r \widehat{\mathbf{h}}_r\Big] = \mathbf{0}_\mathbf{V}.$$

Applying $\mathsf{T}()$ yields that

$$**: \quad \mathbf{0}_\mathbf{H} + \Big[\sum_{r=1}^R \alpha_r \mathbf{h}_r\Big] = \mathbf{0}_\mathbf{H},$$

since each $\mathsf{T}(\mathbf{z}_k) = \mathbf{0}_\mathbf{H}$, and $\mathsf{T}(\widehat{\mathbf{h}}_r) = \mathbf{h}_r$. But $\mathcal{H}$ is LI; consequently all the α-numbers are zero.

Our $(*)$ now says that

$$\sum_{k=1}^N \beta_k \mathbf{z}_k = \mathbf{0}_\mathbf{V}.$$

But $\mathcal{Z}$ is LI, so all β-numbers are zero. QED

Pf: Subspace $\text{Spn}(\mathcal{B})$ is all of $\mathbf{V}$. Consider an arbitrary vector $\mathbf{p} \in \mathbf{V}$. Thus there exist scalars α_r st.

$$\mathsf{T}(\mathbf{p}) = \sum_{r=1}^R \alpha_r \mathbf{h}_r.$$

Define

$$\widehat{\mathbf{p}} := \sum_{r=1}^R \alpha_r \widehat{\mathbf{h}}_r.$$

Since T is linear,

$$\mathsf{T}(\mathbf{p} - \widehat{\mathbf{p}}) = \mathsf{T}(\mathbf{p}) - \mathsf{T}(\widehat{\mathbf{p}}) \stackrel{note}{=} \mathbf{0}_\mathbf{H}.$$

Thus difference-vector $\mathbf{p} - \widehat{\mathbf{p}} \in \text{Ker}(\mathsf{T})$. So there exist scalars β_k for which

$$\sum_{k=1}^N \beta_k \mathbf{z}_k = \mathbf{p} - \widehat{\mathbf{p}}.$$

Hence vector

$$\mathbf{p} \stackrel{note}{=} \Big[\sum_{k=1}^N \beta_k \mathbf{z}_k\Big] + \Big[\sum_{r=1}^R \alpha_r \widehat{\mathbf{h}}_r\Big] \quad ♦$$

is in $\text{Spn}(\mathcal{B})$. QED

The next page has the same proof, but allowing ∞-dim'al spaces. It indexes vectors by themselves, hence is notationally simpler.

Prolegomenon. Here, $\mathbf{V}, \mathbf{H}$ are VSes over a field F, and $\mathsf{T}: \mathbf{V} \to \mathbf{H}$ is linear. Recall

$$\begin{aligned} \text{Nullity}(\mathsf{T}) &:= \text{Dim}(\text{Nul}(\mathsf{T})), \quad \text{and} \\ \text{Rank}(\mathsf{T}) &:= \text{Dim}(\text{Range}(\mathsf{T})), \end{aligned}$$

where these dimensions are (possibly infinite) cardinals.

Recall that a ***linear-combination*** over a set, Ω, of vectors, has form

$$\sum_{\omega \in \Omega} \varphi(\omega) \cdot \omega \,,$$

where function $\varphi: \Omega \to \mathsf{F}$ is ***finitely supported***; that is, the set $\{\omega \in \Omega \mid \varphi(\omega) \neq 0\}$ is finite.

2: *General* Rank-Nullity Thm. *For* $\mathsf{T}: \mathbf{V} \to \mathbf{H}$ *as above,*

$$\text{Rank}(\mathsf{T}) + \text{Nullity}(\mathsf{T}) = \text{Dim}(\text{Dom}(\mathsf{T})) \,. \qquad \diamondsuit$$

Proof. Pick a basis $\mathcal{Z}$ for $\text{Ker}(\mathsf{T})$, and a basis $\mathcal{H}$ for $\text{Range}(\mathsf{T})$. For each $\mathbf{h} \in \mathcal{H}$, choose a vector $\widehat{\mathbf{h}} \in \mathsf{T}^{-1}(\mathbf{h})$. [In principle, this uses the Axiom of Choice.] Define

$$\widehat{\mathcal{H}} := \{\widehat{\mathbf{h}} \mid \mathbf{h} \in \mathcal{H}\} \,.$$

Our goal: The disjoint union $\mathcal{B} := \mathcal{Z} \cup \widehat{\mathcal{H}}$ [of multisets] is a $\mathbf{V}$-basis.

Pf: $\mathcal{B}$ is LI. Consider a linear-combination

$$*: \quad \Big[\sum_{\mathbf{z} \in \mathcal{Z}} \beta(\mathbf{z}) \cdot \mathbf{z}\Big] + \Big[\sum_{\mathbf{h} \in \mathcal{H}} \alpha(\mathbf{h}) \cdot \widehat{\mathbf{h}}\Big] = \mathbf{0}_{\mathbf{V}} \,.$$

[So β and α are each finitely-supported scalar fncs.] Applying $\mathsf{T}()$ yields that

$$**: \quad \mathbf{0}_{\mathbf{H}} + \Big[\sum_{\mathbf{h} \in \mathcal{H}} \alpha(\mathbf{h}) \cdot \mathbf{h}\Big] = \mathbf{0}_{\mathbf{H}} \,,$$

since each $\mathsf{T}(\widehat{\mathbf{h}}) = \mathbf{h}$. But $\mathcal{H}$ is LI; consequently the $\alpha()$ function is identically-zero.

Our $(*)$ now says that

$$\Big[\sum_{\mathbf{z} \in \mathcal{Z}} \beta(\mathbf{z}) \cdot \mathbf{z}\Big] = \mathbf{0}_{\mathbf{V}} \,.$$

But $\mathcal{Z}$ is LI, so $\beta()$ is identically-zero. QED

Pf: Subspace $\text{Spn}(\mathcal{B})$ is all of V. Consider an arbitrary vector $\mathbf{p} \in \mathbf{V}$. Thus there exists $\alpha()$ so that

$$\mathsf{T}(\mathbf{p}) = \sum_{\mathbf{h} \in \mathcal{H}} \alpha(\mathbf{h}) \cdot \mathbf{h} \,.$$

Define

$$\widehat{\mathbf{p}} := \sum_{\mathbf{h} \in \mathcal{H}} \alpha(\mathbf{h}) \cdot \widehat{\mathbf{h}} \,.$$

Because T is linear,

$$\mathsf{T}(\mathbf{p} - \widehat{\mathbf{p}}) = \mathsf{T}(\mathbf{p}) - \mathsf{T}(\widehat{\mathbf{p}}) \stackrel{\text{note}}{=\!=} \mathbf{0}_{\mathbf{H}} \,.$$

Since difference-vector $\mathbf{p} - \widehat{\mathbf{p}} \in \text{Ker}(\mathsf{T})$, there exists a linear-combination $\Big[\sum_{\mathbf{z} \in \mathcal{Z}} \beta(\mathbf{z}) \cdot \mathbf{z}\Big]$ that equals difference $[\mathbf{p} - \widehat{\mathbf{p}}]$. Hence

$$\mathbf{p} \stackrel{\text{note}}{=\!=} \Big[\sum_{\mathbf{z} \in \mathcal{Z}} \beta(\mathbf{z}) \cdot \mathbf{z}\Big] + \Big[\sum_{\mathbf{h} \in \mathcal{H}} \alpha(\mathbf{h}) \cdot \widehat{\mathbf{h}}\Big] \qquad \blacklozenge$$

is in $\text{Spn}(\mathcal{B})$. QED

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