Number Sets. Expression $k \in \mathbb{N}$ [read as "k is an element of $\mathbb{N}$" or "k in $\mathbb{N}$"] means that k is a natural number; a ***natnum***. Expression $\mathbb{N} \ni k$ [read as "$\mathbb{N}$ ***owns*** k"] is a synonym for $k \in \mathbb{N}$.

$\mathbb{N}$ = natural numbers $= \{0, 1, 2, \dots\}$.

$\mathbb{Z}$ = integers $= \{\dots, -2, -1, 0, 1, \dots\}$. For the set $\{1, 2, 3, \dots\}$ of positive integers, the ***posints***, use $\mathbb{Z}_+$. Use $\mathbb{Z}_-$ for the negative integers, the ***negints***.

$\mathbb{Q}$ = rational numbers $= \{\frac{p}{q} \mid p \in \mathbb{Z} \text{ and } q \in \mathbb{Z}_+\}$. Use $\mathbb{Q}_+$ for the positive rationals and $\mathbb{Q}_-$ for the negative rationals.

$\mathbb{R}$ = reals. The ***posreals*** $\mathbb{R}_+$ and the ***negreals*** $\mathbb{R}_-$.

$\mathbb{C}$ = complex numbers, also called the ***complexes***.

For $\omega \in \mathbb{C}$, let "$\omega > 5$" mean "ω *is* ***real*** *and* $\omega > 5$". [Use the same convention for $\geq, <, \leq$, and also if 5 is replaced by any real number.]

Use $\overline{\mathbb{R}} = [-\infty, +\infty] := \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$, the ***extended reals***.

An "***interval of integers***" $[b \,..\, c)$ means the intersection $[b, c) \cap \mathbb{Z}$; ditto for open and closed intervals. So $[\mathrm{e} \,..\, 2\pi] = \{3, 4, 5, 6\} = [3 \,..\, 6] = (2 \,..\, 6]$. We allow b and c to be $\pm\infty$; so $(-\infty \,..\, -1]$ is $\mathbb{Z}_-$. And $[-\infty \,..\, -1]$, is $\{-\infty\} \cup \mathbb{Z}_-$.

Floor function: $\lfloor \pi \rfloor = 3$, $\lfloor -\pi \rfloor = -4$. Ceiling fnc: $\lceil \pi \rceil = 4$. Absolute value: $|-6| = 6 = |6|$ and $|-5 + 2\mathbf{i}| = \sqrt{29}$.

Mathematical objects. Seq: '*sequence*'. poly(s): '*polynomial(s)*'. irred: '*irreducible*'. Coeff: '*coefficient*' and var(s): '*variable(s)*' and parm(s): '*parameter(s)*'. Expr.: '*expression*'. Fnc: '*function*' (so ratfnc: means rational function, a ratio of polynomials). trnfn: '*transformation*'. cty: '*continuity*'. cts: '*continuous*'. diff'able: '*differentiable*'. CoV: '*Change-of-Variable*'. CoI: '*Constant of Integration*'. LoI: '*Limit(s) of Integration*'. RoC: '*Radius of Convergence*'.

Soln: '*Solution*'. Thm: '*Theorem*'. Prop'n: '*Proposition*'. CEX: '*Counterexample*'. eqn: '*equation*'. RhS: '*RightHand side*' of an eqn or inequality. LhS: '*lefthand side*'. Sqrt or Sqroot: '*square-root*', e.g, "the sqroot of 16 is 4". Ptn: '*partition*', *but* pt: '*point*' as in "a fixed-pt of a map".

Binop: '*Binary operator*'. Binrel: '*Binary relation*'.

FTC: '*Fund. Thm of Calculus*'. IVT: '*intermediate-Value Thm*'. MVT: '*Mean-Value Thm*'.

The ***logarithm*** function, defined for $x > 0$, is $\log(x) := \int_1^x \frac{\mathrm{d}v}{v}$. Its inverse-fnc is $\exp()$.

For $x > 0$, then, $\exp(\log(x)) = x = \mathrm{e}^{\log(x)}$. For real t, naturally, $\log(\exp(t)) = t = \log(\mathrm{e}^t)$.

PolyExp: '*Polynomial-times-exponential*', e.g, $[3 + t^2] \cdot \mathrm{e}^{4t}$. PolyExp-sum: '*Sum of polyexps*'. E.g, $f(t) := 3t\mathrm{e}^{2t} + [t^2] \cdot \mathrm{e}^t$ is a polyexp-sum.

Phrases. WLOG: '*Without loss of generality*'. IFF: '*if and only if*'. TFAE: '*The following are equivalent*'. ITOf: '*In Terms Of*'. OTForm: '*of the form*'. FTSOC: '*For the sake of contradiction*'. And ※ = "Contradiction".

IST: '*It Suffices To*', as in ISTShow, ISTExhibit.

Use w.r.t: '*with respect to*' and s.t: '*such that*'.

Latin: e.g: *exempli gratia*, '*for example*'. i.e: *id est*, '*that is*'. N.B: *Nota bene*, '*Note well*'. inter alia: '*among other things*'. QED: *quod erat demonstrandum*, meaning "end of proof".

Bi/Multi-nomial coeffs. For a natnum n, use "$n!$" to mean "n ***factorial***"; the product of all posints $\le n$. So $3! = 3 \cdot 2 \cdot 1 = 6$ and $5! = 120$. Also $0! = 1 = 1!$.

For natnum B and arb. complex number α, define

Rising Fctrl: $[\![\alpha \uparrow \mathsf{B}]\!] := \alpha \cdot [\alpha+1] \cdot [\alpha+2] \cdots [\alpha+[\mathsf{B}-1]]$,
Falling Fctrl: $[\![\alpha \downarrow \mathsf{B}]\!] := \alpha \cdot [\alpha-1] \cdot [\alpha-2] \cdots [\alpha-[\mathsf{B}-1]]$.

E.g, $[\![\mathsf{B}\downarrow\mathsf{B}]\!] = \mathsf{B}! = [\![1\uparrow\mathsf{B}]\!]$. Two further examples,

$$\left[\!\!\left[\tfrac{2}{7} \downarrow 4\right]\!\!\right] = \frac{2}{7} \cdot \frac{-5}{7} \cdot \frac{-12}{7} \cdot \frac{-19}{7} \quad \textit{and} \quad [\![1 \downarrow 3]\!] = 1 \cdot 0 \cdot -1 = 0\,.$$

In particular, for $n \in \mathbb{N}$: If $\mathsf{B} > n$ then $[\![n \downarrow \mathsf{B}]\!] = 0$.

We pronouce $[\![5 \downarrow \mathsf{B}]\!]$ as "5 falling-factorial B".

Binomial. The ***binomial coefficient*** $\binom{7}{3}$, read "7 choose 3", means *the number of ways of choosing 3 objects from 7 distinguishable objects.* Emphasising putting 3 objects in our left pocket and the remaining 4 in our right, we may write the coeff as $\binom{7}{3,4}$. [Read as "7 choose 3-comma-4."] Evidently

$$\dagger: \quad \binom{\mathsf{N}}{j} \overset{\text{with } k:=\mathsf{N}-j}{=\!=\!=} \binom{\mathsf{N}}{j,\ k} = \frac{\mathsf{N}!}{j! \cdot k!} = \frac{[\![\mathsf{N} \downarrow j]\!]}{j!}.$$

Note $\binom{7}{0} = \binom{7}{0,7} = 1$. Finally, the Binomial theorem says

$$\pounds: \qquad [x+y]^{\mathsf{N}} = \sum_{j+k=\mathsf{N}} \binom{\mathsf{N}}{j,k} \cdot x^j y^k\,,$$

where (j,k) ranges over all *ordered* pairs of natural numbers with sum N.

For natnum N, binomials satisfy this addition law:

$$*: \quad \binom{\mathsf{N}+1}{\mathsf{B}+1} = \overbrace{\binom{\mathsf{N}}{\mathsf{B}}}^{\text{Pick last object.}} + \overbrace{\binom{\mathsf{N}}{\mathsf{B}+1}}^{\text{Avoid last object.}}\,.$$

Extending this to *all* $\mathsf{B} \in \mathbb{Z}$ forces:

$$\binom{\mathsf{N}}{\mathsf{B}} = 0, \qquad \textit{when } \mathsf{B} > \mathsf{N} \textit{ or } \mathsf{B} \textit{ negative.}$$

Case $\mathsf{B} > \mathsf{N}$ is automatic in formula $\binom{\mathsf{N}}{\mathsf{B}} = \frac{[\![\mathsf{N}\downarrow\mathsf{B}]\!]}{\mathsf{B}!}$.

Multinomial. In general, for natural numbers $\mathsf{N} = k_1 + \ldots + k_P$, the ***multinomial coefficient*** $\binom{\mathsf{N}}{k_1,\, k_2,\, \ldots,\, k_P}$ is the number of ways of partitioning N objects, by putting k_1 objects in pocket-one, k_2 objects in pocket-two, ...putting k_P objects in the P^{th} pocket. Easily

$$\ddagger: \qquad \binom{\mathsf{N}}{k_1, k_2, \ldots, k_P} = \frac{\mathsf{N}!}{k_1! \cdot k_2! \cdot \ldots \cdot k_P!}\,.$$

Unsurprisingly, $[x_1 + \ldots + x_P]^{\mathsf{N}}$ equals the sum of terms

$$\pounds\pounds: \qquad \binom{\mathsf{N}}{k_1,\ldots,k_P} \cdot {x_1}^{k_1} \cdot {x_2}^{k_2} \cdots {x_P}^{k_P}\,,$$

taken over all natnum-tuples $\vec{k} = (k_1, \ldots, k_P)$ that sum to N. [That is multinomial analog of the Binomial Thm.]

Define the sum $S_\ell := k_1 + k_2 + \ldots + k_\ell$. Then multinomial LhS(‡) equals this product of binomials:

$$\binom{\mathsf{N}}{k_1} \cdot \binom{\mathsf{N}-S_1}{k_2} \cdot \binom{\mathsf{N}-S_2}{k_3} \cdot \ldots \cdot \binom{\mathsf{N}-S_{P-1}}{k_P}.$$

[The last term is $\binom{k_P}{k_P} \overset{\text{note}}{=\!=} 1$.]

Operations on Sets. Use $\in$ for "is an element of". E.g, letting $\mathbb{P}$ be the set of primes, then, $5 \in \mathbb{P}$ yet $6 \notin \mathbb{P}$. Changing the emphasis, $\mathbb{P} \ni 5$ ["$\mathbb{P}$ *owns* 5"] yet $\mathbb{P} \not\ni 6$ ["$\mathbb{P}$ *does-not-own* 6"]

For subsets A and B of the same space, Ω, the ***inclusion relation*** $A \subset B$ means:

$$\forall \omega \in A, \textit{ necessarily } \; B \ni \omega.$$

And this can be written $B \supset A$. Use $A \subsetneqq B$ for *proper* inclusion, i.e, $A \subset B$ yet $A \neq B$.

The *difference set* $B \smallsetminus A$ is $\{\omega \in B \mid \omega \notin A\}$. Employ A^c for the ***complement*** $\Omega \smallsetminus A$. Use $A \triangle B$ for ***symmetric difference*** $[A \smallsetminus B] \cup [B \smallsetminus A]$. Furthermore

$A \boxdot B$,	Sets A & B have *at least* <u>one</u> point in common; they intersect.
$A \sqcap B$,	The sets have *no* common point; disjoint.

The symbol "$A \boxdot B$" both asserts intersection <u>and</u> represents the set $A \cap B$. For a collection $\mathcal{C} = \{E_j\}_j$ of sets in Ω, let the ***disjoint union*** $\bigsqcup_j E_j$ or $\bigsqcup(\mathcal{C})$ represent the union $\bigcup_j E_j$ *and also* asserts that the sets are <u>pairwise</u> <u>disjoint</u>.

See next page...

SeLo [2019t] quizzes so far...

Q1: Fri. 06 Sep $\left[\sqrt{3}^{\sqrt{8}}\right]^{\sqrt{2}} =$ ________. $\log_{64}(16) =$ ________.

Q2: Mon. 09 Sep

May lightning ↯ strike this table! (I.e, please fill in.)

n	r_n	q_n	s_n	t_n
0	100	▬	**1**	**0**
1	23		**0**	**1**
2				
3				
4				
5				

So GCD(100, 23) = ________ = [______ · 100] + [______ · 23].

Q3: Wed. 11 Sep LBolt gives $G \coloneqq \text{GCD}(413, 294) =$ ______. And $413S + 294T = G$, where $S =$ ________ & $T =$ ________ are integers.

Q4: Fri. 13 Sep Binomial coefficient $\binom{7}{4} =$ ____________ = ________.

Q5: Mon. 16 Sep Multinomial coefficient $\binom{9}{4, 2, 3} =$ __________ = ________.

[*Note:* Write your ans. ITOf factorials, then **also** write it as a single integer, or product of two, **without** factorials.]

Q6: Wed. 18 Sep Write the free vars in each of these expressions.

$$\overbrace{\exists n \in \mathbb{N}\colon\ f(n) \subset \underbrace{\bigcup_{\ell=p-4}^{p+7} \{x \in \mathbb{Z} \mid \underbrace{\ell \cdot n \equiv_5 x^2}_{E1}\}}_{E2}}^{E3}$$

$E3$: __________. $E2$: __________. $E1$: __________.

Q7: Fri. 20 Sep Write the truth-table for $B \Rightarrow [[\neg B] \Rightarrow C]$.

B	C	$\neg B$	$[\neg B] \Rightarrow C$	$B \Rightarrow [[\neg B] \Rightarrow C]$
F	F			
F	T			
T	F			
T	T			

Q8: Wed. 02 Oct [In class, we covered the definition of ***convex set***.]

Do one of (a) or (b)

a Sets $\mathbf{A}, \mathbf{B} \subset \mathbb{R}\times\mathbb{R}$ are convex. Prove $\mathbf{A} \cap \mathbf{B}$ is convex.

b *Am I in class today?*

circle one *"Yes!"* *"Of course!"*

"I wouldn't miss it for the world!"

Q9: Fri. 18 Oct *Magic integers* $G_1=$ ________, $G_2=$ ________, $G_3=$ ____________, each in $(-165 \,..\, 165]$, are st. mapping $g\colon \mathbb{Z}_6\times\mathbb{Z}_5\times\mathbb{Z}_{11} \to \mathbb{Z}_{330}$ is a ring-isomorphism, where

$$g((z_1, z_2, z_3)) := \left\langle z_1 G_1 + z_2 G_2 + z_3 G_3 \right\rangle_{330}.$$

Verify for <u>your</u> map: $g((1,1,1)) = 1$ and $[5 \cdot 11] \bullet\!| \; G_1$ and analogously for G_2 and G_3.

QA: Fri. 22 Nov Both $\sim$ and $\bowtie$ are equiv-relations on a set Ω. Define binrels $\mathbf{I}$ and $\mathbf{U}$ on Ω as follows.

Define $\omega \,\mathbf{U}\, \lambda$ IFF *Either* $\omega \sim \lambda$ *or* $\omega \bowtie \lambda$ [or both].
Define $\omega \,\mathbf{I}\, \lambda$ IFF *Both* $\omega \sim \lambda$ *and* $\omega \bowtie \lambda$.

So "$\mathbf{U}$ *is an equiv-relation*" is: T F

So "$\mathbf{I}$ *is an equiv-relation*" is: T F

Let $\mathcal{P}_\infty$ denote the family of all ***co-finite*** subsets of $\mathbb{N}$. That is, a subset $S \subset \mathbb{N}$ is an *element* of $\mathcal{P}_\infty$ IFF $\mathbb{N} \smallsetminus S$ is <u>finite</u>. Define relation $\,_*\!\!\approx^*$ on $\mathcal{P}_\infty$ by: $A \,_*\!\!\approx^*\, B$ IFF $A \cap B$ is infinite.

Stmt "*This* $\,_*\!\!\approx^*$ *is an equivalence-relation*" is: T F

QB: ?? Mon. 25 Nov Solve some of the World's Problems.