Algebra
MAS4301 3249 **Quizzes Q** Friday 26Aug2022

Generalized parentheses.

Common:
( ) *Parentheses.* [ ] *Brackets.* ⟨ ⟩ *Angle-brackets.* { } *Braces.* | | *Vertical bars* (*abs.value, cardinality*).

Less common: ⌊ ⌋ *Floor.* ⌈ ⌉ *Ceiling.* ‖ ‖ *Norm-of.* ⟦ ⟧ *Double-bracket.* ⟨⟨ ⟩⟩ *Double-angle-bracket.*

Number Sets.

Expression $k \in \mathbb{N}$ [read as "k is an element of $\mathbb{N}$" or "k in $\mathbb{N}$"] means that k is a natural number; a ***natnum***. Expression $\mathbb{N} \ni k$ [read as "$\mathbb{N}$ ***owns*** k"] is a synonym for $k \in \mathbb{N}$.

$\mathbb{N}$ = natural numbers $= \{0, 1, 2, \dots\}$.

$\mathbb{Z}$ = integers $= \{\dots, -2, -1, 0, 1, \dots\}$. For the set $\{1, 2, 3, \dots\}$ of positive integers, the ***posints***, use $\mathbb{Z}_+$. Use $\mathbb{Z}_-$ for the negative integers, the ***negints***.

$\mathbb{Q}$ = rational numbers $= \{\frac{p}{q} \mid p \in \mathbb{Z} \text{ and } q \in \mathbb{Z}_+\}$. Use $\mathbb{Q}_+$ for the positive rationals and $\mathbb{Q}_-$ for the negative rationals.

$\mathbb{R}$ = reals. The ***posreals*** $\mathbb{R}_+$ and the ***negreals*** $\mathbb{R}_-$.

$\mathbb{C}$ = complex numbers, also called the ***complexes***.

For $\omega \in \mathbb{C}$, let "$\omega > 5$" mean "ω *is* **real** *and* $\omega > 5$". [Use the same convention for $\geq, <, \leq$, and also if 5 is replaced by any real number.]

Use $\overline{\mathbb{R}} = [-\infty, +\infty] := \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$, the ***extended reals***.

An "***interval of integers***" $[b \,..\, c)$ means the intersection $[b, c) \cap \mathbb{Z}$; ditto for open and closed intervals. So $[\mathrm{e} \,..\, 2\pi] = \{3, 4, 5, 6\} = [3 \,..\, 6] = (2 \,..\, 6]$. We allow b and c to be $\pm\infty$; so $(-\infty \,..\, -1]$ is $\mathbb{Z}_-$. And $[-\infty \,..\, -1]$, is $\{-\infty\} \cup \mathbb{Z}_-$.

Floor function: $\lfloor \pi \rfloor = 3$, $\lfloor -\pi \rfloor = -4$. Ceiling fnc: $\lceil \pi \rceil = 4$. Absolute value: $|-6| = 6 = |6|$ and $|-5 + 2\mathbf{i}| = \sqrt{29}$.

Mathematical objects.

Seq: '*sequence*'. poly(s): '*polynomial(s)*'. irred: '*irreducible*'. Coeff: '*coefficient*' and var(s): '*variable(s)*' and parm(s): '*parameter(s)*'. Expr.: '*expression*'. Fnc: '*function*' (so ratfnc: means rational function, a ratio of polynomials). trnfn: '*transformation*'. cty: '*continuity*'. cts: '*continuous*'. diff'able: '*differentiable*'. CoV: '*Change-of-Variable*'. CoI: '*Constant of Integration*'. LoI: '*Limit(s) of Integration*'. RoC: '*Radius of Convergence*'.

Soln: '*Solution*'. Thm: '*Theorem*'. Prop'n: '*Proposition*'. CEX: '*Counterexample*'. eqn: '*equation*'. RhS: '*RightHand side*' of an eqn or inequality. LhS: '*lefthand side*'. Sqrt or Sqroot: '*square-root*', e.g, "the sqroot of 16 is 4". Ptn: '*partition*', *but* pt: '*point*' as in "a fixed-pt of a map".

Binop: '*Binary operator*'. Binrel: '*Binary relation*'.

FTC: '*Fund. Thm of Calculus*'. IVT: '*intermediate-Value Thm*'. MVT: '*Mean-Value Thm*'.

The ***logarithm*** function, defined for $x>0$, is $\log(x) := \int_1^x \frac{dv}{v}$. Its inverse-fnc is $\exp()$.

For $x>0$, then, $\exp(\log(x)) = x = \mathrm{e}^{\log(x)}$. For real t, naturally, $\log(\exp(t)) = t = \log(\mathrm{e}^t)$.

PolyExp: '*Polynomial-times-exponential*', e.g, $[3 + t^2] \cdot \mathrm{e}^{4t}$. PolyExp-sum: '*Sum of polyexps*'. E.g, $f(t) := 3t\mathrm{e}^{2t} + [t^2] \cdot \mathrm{e}^t$ is a polyexp-sum.

Phrases.

WLOG: '*Without loss of generality*'. IFF: '*if and only if*'. TFAE: '*The following are equivalent*'. ITOf: '*In Terms Of*'. OTForm: '*of the form*'. FTSOC: '*For the sake of contradiction*'. And ※ ="Contradiction".

IST: '*It Suffices To*', as in ISTShow, ISTExhibit.

Use w.r.t: '*with respect to*' and s.t: '*such that*'.

Latin: e.g: *exempli gratia*, '*for example*'. i.e: *id est*, '*that is*'. N.B: *Nota bene*, '*Note well*'. inter alia: '*among other things*'. QED: *quod erat demonstrandum*, meaning "end of proof".

Bi/Multi-nomial coeffs. For a natnum n, use "$n!$" to mean "n ***factorial***"; the product of all posints $\le n$. So $3! = 3 \cdot 2 \cdot 1 = 6$ and $5! = 120$. Also $0! = 1 = 1!$.

For natnum B and arb. complex number α, define

Rising Fctrl: $[\![\alpha \uparrow \mathsf{B}]\!] := \alpha \cdot [\alpha+1] \cdot [\alpha+2] \cdots [\alpha + [\mathsf{B}-1]]$,
Falling Fctrl: $[\![\alpha \downarrow \mathsf{B}]\!] := \alpha \cdot [\alpha-1] \cdot [\alpha-2] \cdots [\alpha - [\mathsf{B}-1]]$.

E.g, $[\![\mathsf{B}\downarrow\mathsf{B}]\!] = \mathsf{B}! = [\![1\uparrow\mathsf{B}]\!]$. Two further examples,

$$\left[\!\!\left[\tfrac{2}{7} \downarrow 4\right]\!\!\right] = \frac{2}{7} \cdot \frac{-5}{7} \cdot \frac{-12}{7} \cdot \frac{-19}{7} \quad \textit{and} \quad [\![1 \downarrow 3]\!] = 1 \cdot 0 \cdot -1 = 0\,.$$

In particular, for $n \in \mathbb{N}$: If $\mathsf{B} > n$ then $[\![n \downarrow \mathsf{B}]\!] = 0$. We pronouce $[\![5 \downarrow \mathsf{B}]\!]$ as "5 falling-factorial B".

Binomial. The ***binomial coefficient*** $\binom{7}{3}$, read "7 choose 3", means *the number of ways of choosing 3 objects from 7 distinguishable objects.* Emphasising putting 3 objects in our left pocket and the remaining 4 in our right, we may write the coeff as $\binom{7}{3,4}$. [Read as "7 choose 3-comma-4."] Evidently

$$\dagger: \quad \binom{\mathsf{N}}{j} \xlongequal{\text{with } k := \mathsf{N}-j} \binom{\mathsf{N}}{j,\ k} = \frac{\mathsf{N}!}{j! \cdot k!} = \frac{[\![\mathsf{N} \downarrow j]\!]}{j!}.$$

Note $\binom{7}{0} = \binom{7}{0,7} = 1$. Finally, the Binomial theorem says

$$\pounds: \qquad [x+y]^{\mathsf{N}} = \sum_{j+k=\mathsf{N}} \binom{\mathsf{N}}{j,k} \cdot x^j y^k\,,$$

where (j,k) ranges over all *ordered* pairs of natural numbers with sum N.

For natnum N, binomials satisfy this addition law:

$$*: \quad \binom{\mathsf{N}+1}{\mathsf{B}+1} = \overbrace{\binom{\mathsf{N}}{\mathsf{B}}}^{\text{Pick last object.}} + \overbrace{\binom{\mathsf{N}}{\mathsf{B}+1}}^{\text{Avoid last object.}}\,.$$

Extending this to *all* $\mathsf{B} \in \mathbb{Z}$ forces:

$$\binom{\mathsf{N}}{\mathsf{B}} = 0, \quad \begin{array}{l}\textit{when } \mathsf{B} > \mathsf{N} \\ \textit{or } \mathsf{B} \textit{ negative.}\end{array}$$

Case $\mathsf{B}>\mathsf{N}$ is automatic in formula $\binom{\mathsf{N}}{\mathsf{B}} = \frac{[\![\mathsf{N}\downarrow\mathsf{B}]\!]}{\mathsf{B}!}$.

Multinomial. In general, for natural numbers $\mathsf{N} = k_1 + \ldots + k_P$, the ***multinomial coefficient*** $\binom{\mathsf{N}}{k_1,\,k_2,\,\ldots,\,k_P}$ is the number of ways of partitioning N objects, by putting k_1 objects in pocket-one, k_2 objects in pocket-two, ...putting k_P objects in the P^{th} pocket. Easily

$$\ddagger: \qquad \binom{\mathsf{N}}{k_1, k_2, \ldots, k_P} = \frac{\mathsf{N}!}{k_1! \cdot k_2! \cdot \ldots \cdot k_P!}\,.$$

Unsurprisingly, $[x_1 + \ldots + x_P]^{\mathsf{N}}$ equals the sum of terms

$$\pounds\pounds: \qquad \binom{\mathsf{N}}{k_1,\ldots,k_P} \cdot {x_1}^{k_1} \cdot {x_2}^{k_2} \cdots {x_P}^{k_P}\,,$$

taken over all natnum-tuples $\vec{k} = (k_1, \ldots, k_P)$ that sum to N. [That is multinomial analog of the Binomial Thm.]

Define the sum $S_\ell := k_1 + k_2 + \ldots + k_\ell$. Then multinomial LhS(‡) equals this product of binomials:

$$\binom{\mathsf{N}}{k_1} \cdot \binom{\mathsf{N} - S_1}{k_2} \cdot \binom{\mathsf{N} - S_2}{k_3} \cdot \ldots \cdot \binom{\mathsf{N} - S_{P-1}}{k_P}.$$

[The last term is $\binom{k_P}{k_P} \xlongequal{\text{note}} 1$.]

Operations on Sets. Use $\in$ for "is an element of". E.g, letting $\mathbb{P}$ be the set of primes, then, $5 \in \mathbb{P}$ yet $6 \notin \mathbb{P}$. Changing the emphasis, $\mathbb{P} \ni 5$ ["$\mathbb{P}$ *owns* 5"] yet $\mathbb{P} \not\ni 6$ ["$\mathbb{P}$ *does-not-own* 6"]

For subsets A and B of the same space, Ω, the ***inclusion relation*** $A \subset B$ means:

$$\forall \omega \in A, \textit{ necessarily } \; B \ni \omega.$$

And this can be written $B \supset A$. Use $A \subsetneqq B$ for *proper* inclusion, i.e, $A \subset B$ yet $A \neq B$.

The *difference set* $B \smallsetminus A$ is $\{\omega \in B \mid \omega \notin A\}$. Employ A^c for the ***complement*** $\Omega \smallsetminus A$. Use $A \triangle B$ for ***symmetric difference*** $[A \smallsetminus B] \cup [B \smallsetminus A]$. Furthermore

$A \boxdot B$,	Sets A & B have *at least* one point in common; they intersect.
$A \sqcap B$,	The sets have *no* common point; disjoint.

The symbol "$A \boxdot B$" both asserts intersection and represents the set $A \cap B$. For a collection $\mathcal{C} = \{E_j\}_j$ of sets in Ω, let the ***disjoint union*** $\bigsqcup_j E_j$ or $\bigsqcup(\mathcal{C})$ represent the union $\bigcup_j E_j$ *and also* asserts that the sets are pairwise disjoint.

See next page...

Algebra [2022t] quizzes so far...

Q1: Fri. 09 Sep

May lightning ↯ strike this table! (I.e, please fill in.)

n	r_n	q_n	s_n	t_n
0	100	—	**1**	**0**
1	23		**0**	**1**
2				
3				
4				
5				

So $\text{GCD}(100, 23) = \underline{\qquad} = [\underline{\qquad} \cdot 100] + [\underline{\qquad} \cdot 23]$.

And $x = \underline{\qquad} \in (-50 \,..\, 50]$ solves congr. $23x \equiv_{100} 8$.

Q2: Mon. 12 Sep Dihedral group $\mathbb{D}_{12}$, the symmetries of a dodecagon, is generated by a rotation R and a F. Product

$$\mathtt{F}\,\mathtt{R}^2\,\mathtt{F}^2\,\mathtt{R}^{-3}\,\mathtt{F}\,\mathtt{R}^{-5}\,\mathtt{F}^3\,\mathtt{R}^7 = \mathtt{R}^J\,\mathtt{F}^K,$$

where $J = \underline{\qquad} \in [0 \,..\, 12)$ and $K = \underline{\qquad} \in \{0, 1\}$.

Q3: Wed. 14 Sep Euler $\varphi(36300) = 2^A \cdot 3^B \cdot 5^C \cdot 7^D \cdot 11^E$, where $A = \underline{\qquad}$, $B = \underline{\qquad}$, $C = \underline{\qquad}$, $D = \underline{\qquad}$, $E = \underline{\qquad}$ are in $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$.

Q4: Fri. 18 Nov Consider groups $A \supset B \supset C$.

If $A \triangleright B$ and $A \triangleright C$ then $B \triangleright C$: *T* *F*

If $A \triangleright C$ and $B \triangleright C$ then $A \triangleright B$: *T* *F*

If $A \triangleright B$ and $B \triangleright C$ then $A \triangleright C$: *T* *F*

Q5: Mon. 21 Nov Counting elements in groups, $|\mathbb{S}_5| =$ .... and $|\mathbb{A}_5| =$ ....

Q6: Mon. 28 Nov Three groups satisfy $\mathsf{N} \triangleleft G$ and $\mathsf{K} \triangleleft G$. *Then...*

Subgroup $[\mathsf{N} \cap \mathsf{K}] \triangleleft G$: *T* *F*

Subgroup $\langle \mathsf{N}, \mathsf{K} \rangle_G \triangleleft G$: *T* *F*

Q7: Mon. 05 Dec *EoS 2022 Games Party*, from 11:45am–4pm, will take place at *Pascal's Cafe* on Wedn. 7Dec.

Circle **Yes!** **True!** **I-have-a-game!**

Games Party: !!

Wed. 07 Dec Bring games and look photogenic, for our traditional Games Party, from 11:45 am to 4 pm, at Pascal's Cafe.