Number Sets. Expression $k \in \mathbb{N}$ [read as "k is an element of $\mathbb{N}$" or "k in $\mathbb{N}$"] means that k is a natural number; a **natnum**. Expression $\mathbb{N} \ni k$ [read as "$\mathbb{N}$ **owns** k"] is a synonym for $k \in \mathbb{N}$.

$\mathbb{N}$ = natural numbers = $\{0, 1, 2, \dots\}$.

$\mathbb{Z}$ = integers = $\{\dots, -2, -1, 0, 1, \dots\}$. For the set $\{1, 2, 3, \dots\}$ of positive integers, the **posints**, use $\mathbb{Z}_+$. Use $\mathbb{Z}_-$ for the negative integers, the **negints**.

$\mathbb{Q}$ = rational numbers = $\{\frac{p}{q} \mid p \in \mathbb{Z} \text{ and } q \in \mathbb{Z}_+\}$. Use $\mathbb{Q}_+$ for the positive rationals and $\mathbb{Q}_-$ for the negative rationals.

$\mathbb{R}$ = reals. The **posreals** $\mathbb{R}_+$ and the **negreals** $\mathbb{R}_-$. $\mathbb{C}$ = complex numbers, also called the **complexes**.

For $\omega \in \mathbb{C}$, let "$\omega > 5$" mean "ω is **real** <u>and</u> $\omega > 5$". [Use the same convention for $\geq, <, \leq$, and also if 5 is replaced by any real number.]

Use $\overline{\mathbb{R}} = [-\infty, +\infty] := \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$, the **extended reals**.

An "**interval of integers**" $[b \mathinner{..} c)$ means the intersection $[b, c) \cap \mathbb{Z}$; ditto for open and closed intervals. So $[e \mathinner{..} 2\pi] = \{3, 4, 5, 6\} = [3 \mathinner{..} 6] = (2 \mathinner{..} 6]$. We allow b and c to be $\pm\infty$; so $(-\infty \mathinner{..} -1]$ is $\mathbb{Z}_-$. And $[-\infty \mathinner{..} -1]$, is $\{-\infty\} \cup \mathbb{Z}_-$.

Floor function: $\lfloor \pi \rfloor = 3$, $\lfloor -\pi \rfloor = -4$. Ceiling fnc: $\lceil \pi \rceil = 4$. Absolute value: $|-6| = 6 = |6|$ and $|-5 + 2\mathbf{i}| = \sqrt{29}$.

Mathematical objects. Seq: '*sequence*'. poly(s): '*polynomial(s)*'. irred: '*irreducible*'. Coeff: '*coefficient*' and var(s): '*variable(s)*' and parm(s): '*parameter(s)*'. Expr.: '*expression*'. Fnc: '*function*' (so ratfnc: means rational function, a ratio of polynomials). trnfn: '*transformation*'. cty: '*continuity*'. cts: '*continuous*'. diff'able: '*differentiable*'. CoV: '*Change-of-Variable*'. CoI: '*Constant of Integration*'. LoI: '*Limit(s) of Integration*'. RoC: '*Radius of Convergence*'.

Soln: '*Solution*'. Thm: '*Theorem*'. Prop'n: '*Proposition*'. CEX: '*Counterexample*'. eqn: '*equation*'. RhS: '*RightHand side*' of an eqn or inequality. LhS: '*lefthand side*'. Sqrt **or** Sqroot: '*square-root*', e.g, "the sqroot of 16 is 4". Ptn: '*partition*', <u>but</u> pt: '*point*' as in "a fixed-pt of a map".

Binop: '*Binary operator*'. Binrel: '*Binary relation*'.

FTC: '*Fund. Thm of Calculus*'. IVT: '*intermediate-Value Thm*'. MVT: '*Mean-Value Thm*'.

The **logarithm** function, defined for $x > 0$, is $\log(x) := \int_1^x \dfrac{\mathrm{d}v}{v}$. Its inverse-fnc is $\exp()$.

For $x > 0$, then, $\exp(\log(x)) = x = \mathrm{e}^{\log(x)}$. For real t, naturally, $\log(\exp(t)) = t = \log(\mathrm{e}^t)$.

PolyExp: '*Polynomial-times-exponential*',   e.g, $[3 + t^2] \cdot \mathrm{e}^{4t}$.   PolyExp-sum: '*Sum of polyexps*'. E.g, $f(t) := 3t\mathrm{e}^{2t} + [t^2] \cdot \mathrm{e}^t$ is a polyexp-sum.

Phrases. WLOG: '*Without loss of generality*'. IFF: '*if and only if*'. TFAE: '*The following are equivalent*'. ITOf: '*In Terms Of*'. OTForm: '*of the form*'. FTSOC: '*For the sake of contradiction*'. And ※ ="Contradiction". IST: '*It Suffices To*', as in ISTShow, ISTExhibit.

Use w.r.t: '*with respect to*' and s.t: '*such that*'.

Latin: e.g: *exempli gratia*, '*for example*'. i.e: *id est*, '*that is*'. N.B: *Nota bene*, '*Note well*'. inter alia: '*among other things*'. QED: *quod erat demonstrandum*, meaning "end of proof".

Bi/Multi-nomial coeffs. For a natnum n, use "$n!$" to mean "n **factorial**"; the product of all posints $\leq n$. So $3! = 3 \cdot 2 \cdot 1 = 6$ and $5! = 120$. Also $0! = 1 = 1!$.

For natnum B and arb. complex number α, define

Rising Fctrl: $[\![\alpha \uparrow \mathsf{B}]\!] := \alpha \cdot [\alpha + 1] \cdot [\alpha + 2] \cdots [\alpha + [\mathsf{B}-1]]$,

Falling Fctrl: $[\![\alpha \downarrow \mathsf{B}]\!] := \alpha \cdot [\alpha - 1] \cdot [\alpha - 2] \cdots [\alpha - [\mathsf{B}-1]]$.

E.g, $[\![\mathsf{B} \downarrow \mathsf{B}]\!] = \mathsf{B}! = [\![1 \uparrow \mathsf{B}]\!]$. Two further examples,

$$\left[\!\!\left[\tfrac{2}{7} \downarrow 4\right]\!\!\right] = \frac{2}{7} \cdot \frac{-5}{7} \cdot \frac{-12}{7} \cdot \frac{-19}{7} \text{ and } [\![1 \downarrow 3]\!] = 1 \cdot 0 \cdot -1 = 0.$$

In particular, for $n \in \mathbb{N}$: If $\mathsf{B} > n$ then $[\![n \downarrow \mathsf{B}]\!] = 0$. We pronouce $[\![5 \downarrow \mathsf{B}]\!]$ as "5 falling-factorial B".

Binomial. The **binomial coefficient** $\binom{7}{3}$, read "7 choose 3", means *the number of ways of choosing 3 objects from 7 distinguishable objects*. Emphasising putting 3 objects in our left pocket and the remaining 4 in our right, we may write the coeff as $\binom{7}{3,4}$. [Read as "7 choose 3-comma-4."] Evidently

$$\dagger: \qquad \binom{\mathsf{N}}{j} \overset{\text{with } k := \mathsf{N} - j}{=\!=\!=\!=\!=} \binom{\mathsf{N}}{j,\ k} = \frac{\mathsf{N}!}{j! \cdot k!} = \frac{[\![\mathsf{N} \downarrow j]\!]}{j!}.$$

Note $\binom{7}{0} = \binom{7}{0,7} = 1$. Finally, the Binomial theorem says

$$\pounds: \qquad [x + y]^{\mathsf{N}} = \sum_{j+k=\mathsf{N}} \binom{\mathsf{N}}{j,k} \cdot x^j y^k,$$

where (j, k) ranges over all *ordered* pairs of natural numbers with sum N.

For natnum N, binomials satisfy this addition law:

$$\ast: \quad \binom{\mathsf{N}+1}{\mathsf{B}+1} = \overbrace{\binom{\mathsf{N}}{\mathsf{B}}}^{\text{Pick last object.}} + \overbrace{\binom{\mathsf{N}}{\mathsf{B}+1}}^{\text{Avoid last object.}}.$$

Extending this to *all* $\mathsf{B} \in \mathbb{Z}$ forces:

$$\binom{\mathsf{N}}{\mathsf{B}} = 0, \quad \begin{array}{l} \textit{when } \mathsf{B} > \mathsf{N} \\ \textit{or } \mathsf{B} \textit{ negative.} \end{array}$$

Case $\mathsf{B} > \mathsf{N}$ is automatic in formula $\binom{\mathsf{N}}{\mathsf{B}} = \frac{[\![\mathsf{N} \downarrow \mathsf{B}]\!]}{\mathsf{B}!}$.

Multinomial. In general, for natural numbers $\mathsf{N} = k_1 + \ldots + k_P$, the **multinomial coefficient** $\binom{\mathsf{N}}{k_1,\, k_2,\, \ldots,\, k_P}$ is the number of ways of partitioning N objects, by putting k_1 objects in pocket-one, k_2 objects in pocket-two, ... putting k_P objects in the Pth pocket. Easily

$$\ddagger: \qquad \binom{\mathsf{N}}{k_1, k_2, \ldots, k_P} = \frac{\mathsf{N}!}{k_1! \cdot k_2! \cdot \ldots \cdot k_P!}.$$

Unsurprisingly, $[x_1 + \ldots + x_P]^{\mathsf{N}}$ equals the sum of terms

$$\pounds\pounds: \qquad \binom{\mathsf{N}}{k_1, \ldots, k_P} \cdot x_1^{k_1} \cdot x_2^{k_2} \cdots x_P^{k_P},$$

taken over all natnum-tuples $\vec{k} = (k_1, \ldots, k_P)$ that sum to N. [That is multinomial analog of the Binomial Thm.]

Define the sum $S_\ell := k_1 + k_2 + \ldots + k_\ell$. Then multinomial LhS($\ddagger$) equals this product of binomials:

$$\binom{\mathsf{N}}{k_1} \cdot \binom{\mathsf{N} - S_1}{k_2} \cdot \binom{\mathsf{N} - S_2}{k_3} \cdot \ldots \cdot \binom{\mathsf{N} - S_{P-1}}{k_P}.$$

[The last term is $\binom{k_P}{k_P} \overset{\text{note}}{=\!=} 1$.]

Operations on Sets. Use $\in$ for "is an element of". E.g, letting $\mathbb{P}$ be the set of primes, then, $5 \in \mathbb{P}$ yet $6 \notin \mathbb{P}$. Changing the emphasis, $\mathbb{P} \ni 5$ ["$\mathbb{P}$ *owns* 5"] yet $\mathbb{P} \not\ni 6$ ["$\mathbb{P}$ *does-not-own* 6"]

For subsets A and B of the same space, Ω, the **inclusion relation** $A \subset B$ means:

$$\forall \omega \in A, \textit{ necessarily } B \ni \omega.$$

And this can be written $B \supset A$. Use $A \subsetneqq B$ for *proper* inclusion, i.e, $A \subset B$ yet $A \neq B$.

The *difference set* $B \smallsetminus A$ is $\{\omega \in B \mid \omega \notin A\}$. Employ A^c for the **complement** $\Omega \smallsetminus A$. Use $A \triangle B$ for **symmetric difference** $[A \smallsetminus B] \cup [B \smallsetminus A]$. Furthermore

$$A \,\boxdot\, B, \qquad \text{Sets } A \,\&\, B \text{ have } \textit{at least } \underline{\text{one}} \text{ point in common; they intersect.}$$

$$A \,\sqcap\, B, \qquad \text{The sets have } \textit{no} \text{ common point; disjoint.}$$

The symbol "$A \boxdot B$" both asserts intersection <u>and</u> represents the set $A \cap B$. For a collection $\mathcal{C} = \{E_j\}_j$ of sets in Ω, let the **disjoint union** $\bigsqcup_j E_j$ or $\bigsqcup(\mathcal{C})$ represent the union $\bigcup_j E_j$ *and also* asserts that the sets are <u>pairwise</u> <u>disjoint</u>.

See next page...

Algebra [2019t] quizzes so far...

P1: $^{\text{Mon.}}_{\text{09 Sep}}$ May lightning $\nmid$ strike this table! (I.e, please fill in.)

n	r_n	q_n	s_n	t_n
0	174	—	1	0
1	51		0	1
2				
3				
4				
5				

So GCD$(174, 51)=$ ⌊....⌋ $=[$ ⌊........⌋ $\cdot 174] + [$ ⌊........⌋ $\cdot 51]$.

P2: $^{\text{Mod.}}_{\text{16 Sep}}$ Dihedral group $\mathbb{D}_{12}$, the symmetries of a dodecagon, is generated by a rotation R and a F. Product

$$\texttt{F R}^2 \texttt{ F}^2 \texttt{ R}^{-3} \texttt{ F R}^{-5} \texttt{ F}^3 \texttt{ R}^7 \;=\; \texttt{R}^J \texttt{F}^K \,,$$

where $J =$ ⌊.....⌋ $\in [0 .. 12)$ and $K =$ ⌊...⌋ $\in \{0, 1\}$.

P3: $^{\text{Fri.}}_{\text{27 Sep}}$ Euler $\varphi(121000) =$ ⌊....................⌋ .

Express your answer as a product $p_1{}^{e_1} \cdot p_2{}^{e_2} \cdot \ldots$ of *primes* to posint powers, with $p_1 < p_2 < \ldots$.

P4: $^{\text{Wed.}}_{\text{02 Oct}}$ Perm $\gamma \in \mathbb{S}_{10}$ is $\langle 4\,3\,2\,1\,0 \rangle \langle 9\,8\,7\,6\,5 \rangle$.

The possible cycle-types of β, where $\beta^2 = \gamma$, are

.... And our γ

has .... many sqroots.

P5: $^{\text{Fri.}}_{\text{11 Oct}}$ *Am I in class today?*

[circle one]   *"Yes!"*   *"Of course!"*

"I wouldn't miss it for the world!"

P6: $^{\text{Wed.}}_{\text{16 Oct}}$ $\#\mathrm{Inn}(\mathbb{D}_{10}) = $ .... and $\#\mathrm{Aut}(\mathbb{D}_{10}) = $ ..... .

P7: $^{\text{Mod.}}_{\text{04 Nov}}$ In $\mathbb{S}_4$, define $\alpha := \langle 1,2,3 \rangle$ and $\beta := \langle 3,4 \rangle$. Compute commutator $[\![\alpha, \beta]\!] := \alpha\beta\alpha^{-1}\beta^{-1} = $ ....

P8: $^{\text{Fri.}}_{\text{22 Nov}}$ The OP-isometry group, G, of the tetrahedron, **T**, has $|G| = $ .... K-coloring the edges of **T**, there are

.... $\cdot \left[K^6 + \right.$ .... $\left.\right]$ G-distinct colorings.

[Burnside's thm, P.474.]

P9: $\overset{!\,!}{\cdot}$ $^{\text{Mon.}}_{\text{25 Nov}}$ Solve some of the World's Problems.

P: $^{\text{Wed.}}_{\text{27 Nov}}$ *No class!*