

Gauss's Quadratic Reciprocity Theorem : NumThy

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1: Nomenclature. For odd D , use H_D to mean $\frac{D-1}{2}$. (The H is to suggest “Half”.)

In the sequel, p is oddprime and $S \perp p$ is the “stride-length”; we will walk around the circumference= p circle using strides of length S .

Use $H := H_p$ and $\langle x \rangle := \langle x \rangle_p$ for the *symmetric* residue of integer x modulo p ; so $\langle x \rangle$ is in $[-H .. H]$.
Let \equiv mean \equiv_p .

Let $\mathcal{G} = \mathcal{G}_p(S)$ be the set of indices $\ell \in [1 .. H]$ such that $\langle \ell \cdot S \rangle_p$ is neGative. Letting \mathcal{P} be the indices with $\langle \ell \cdot S \rangle$ Positive, we have that (disjointly)

$$\mathcal{G} \sqcup \mathcal{P} = [1 .. H]. \quad (\text{The “Time” set.})$$

Finally, use $\mathcal{N} = \mathcal{N}_p(S)$ for the number of “negative” indices; $\mathcal{N} := \#\mathcal{G}$. □

2: Prop'n. Fix an $S \perp p$, with notation from (1). Then the mapping [absolute-value of symm-residue]

$$\ell \mapsto |\langle \ell \cdot S \rangle|,$$

is a **permutation** of $[1 .. H]$. Mapping $\ell \mapsto \langle \ell \cdot S \rangle$ is a “permutation up to sign” of $[1 .. H]$. ◇

Proof. Given indices $1 \leq \mathbf{k} \leq \ell \leq H$, we want that either equality $\mp \langle \mathbf{k} \cdot S \rangle = \langle \ell \cdot S \rangle$ forces $\mathbf{k} = \ell$.

For either choice of sign in \mp , note that

$$\begin{aligned} \mp \langle \mathbf{k} \cdot S \rangle = \langle \ell \cdot S \rangle & \quad \text{IFF} \quad 0 \equiv [\ell \pm \mathbf{k}] \cdot S \\ & \quad \text{IFF} \quad 0 \equiv \ell \pm \mathbf{k}, \end{aligned}$$

since $S \perp p$. Thus

$$0 \leq \ell \pm \mathbf{k} \leq 2H < p.$$

Together with $\ell \pm \mathbf{k} \equiv 0$, this forces $\ell \pm \mathbf{k}$ to actually be zero. Thus the “ \pm ” is a minus sign, and $\ell = \mathbf{k}$. ◇

3: Gauss Lemma. Fix an odd prime p and integer $S \perp p$. Then the Legendre symbol $\left(\frac{S}{p}\right)$ satisfies

$$\left(\frac{S}{p}\right) = [-1]^{\mathcal{N}_p}. \quad \diamond$$

Pf of Gauss Lemma. Let $\mathcal{N} := \mathcal{N}_p(S)$. Necessarily

$$*: \prod_{\ell=1}^H \langle \ell \cdot S \rangle \equiv \prod_{\ell=1}^H \ell \cdot S = H! \cdot S^H \equiv H! \cdot \left(\frac{S}{p}\right),$$

with the last step following from LSThm. Observe that $\langle \ell \cdot S \rangle$ equals $\pm |\langle \ell S \rangle|$ as ℓ is-not/is in \mathcal{G} . Prop'n 2, consequently, tells us that LhS(*) can be written as $H!$ times $[-1]^{\mathcal{N}}$. Thus RhS(*) equals

$$H! \cdot \left(\frac{S}{p}\right) \equiv H! \cdot [-1]^{\mathcal{N}}.$$

The $H!$, being co-prime to p , cancels mod- p to hand us congruence $\left(\frac{S}{p}\right) \equiv [-1]^{\mathcal{N}}$. ◇

An important application is the following.

4: Two-is-QR Lemma. Consider an oddprime p . Then 2 is a p -QR IFF $p \equiv_8 \pm 1$. ◇

Abbrev. An odd integer D is **8Near** if $D \equiv_8 \pm 1$; it is **8Far** if $D \equiv_8 \pm 3$. [The names come from being, mod 8, near/far from zero.] □

Proof. Call p “good” if 2 is a p -QR. As usual, let $H := \frac{p-1}{2}$. It is easy to check that

$$\begin{aligned} \dagger: \quad \text{Even } H & \iff p \equiv_8 \{+1, -3\}; \\ \ddagger: \quad \text{Odd } H & \iff p \equiv_8 \{-1, +3\}. \end{aligned}$$

Let $\mathcal{G} := \mathcal{G}_p(2)$. Computing $\mathcal{N} := |\mathcal{G}|$ has two cases:

CASE: H is even Here, $\mathcal{N} = [H - \frac{H+2}{2}] + 1 = \frac{H}{2}$, since $\mathcal{G} = \{\frac{H+2}{2}, \frac{H+3}{2}, \dots, H\}$.

CASE: H is odd So $\mathcal{N} = [H - \frac{H+1}{2}] + 1 = \frac{H+1}{2}$, since $\mathcal{G} = \{\frac{H+1}{2}, \frac{H+2}{2}, \dots, H\}$.

The Gauss lemma directs us to examine $\mathcal{N} \bmod 2$.

CASE: H is even. Courtesy (\dagger) we can write p as $8L + \{1, -3\}$, with $L \in \mathbb{Z}$. Thus

$$H = \frac{8L + \{1, -3\} - 1}{2} = 4L + \{0, -2\}.$$

So $\mathcal{N} = \frac{H}{2} = 2L + \{0, -1\}$. Consequently,

$$p \text{ good} \iff \mathcal{N} \equiv_2 0 \iff H \equiv_4 0 \iff p \equiv_8 1.$$

CASE: H is odd. We can write $p = 8L + \{-1, +3\}$.
Thus

$$H+1 = \frac{8L + \{-1, +3\} - 1}{2} + 1 = 4L + \{0, 2\}.$$

So $\mathcal{N} = \frac{H+1}{2} = 2L + \{0, 1\}$. Consequently,

$$p \text{ good} \iff \mathcal{N} \equiv_2 0 \iff H+1 \equiv_4 0 \iff p \equiv_8 -1.$$

This gives the lemma. \blacklozenge

The Wrapping function. Count “full **Wraps**”,

$$\mathcal{W} = \mathcal{W}_p(S) := \sum_{\ell=1}^{H_p} \left\lfloor \frac{\ell \cdot S}{p} \right\rfloor,$$

when walking around the circle with stridlength S .
(Here, $\lfloor \cdot \rfloor$ is the floor function.)

5: Eisenstein Lemma. Fix $S \perp p$ from (1), with S odd.
Then \mathcal{N} and \mathcal{W} are either both even or both odd. I.e.

$$\mathcal{N}_p(S) \equiv_2 \mathcal{W}_p(S). \quad \blacklozenge$$

Proof of Eisenstein Lemma. Let $r_\ell := \langle \ell \cdot S \rangle_p$, Then

$$\ell \cdot S = p \cdot \left\lfloor \frac{\ell \cdot S}{p} \right\rfloor + \begin{cases} r_\ell & \text{if } \ell \in \mathcal{P} \\ p + r_\ell & \text{if } \ell \in \mathcal{G} \end{cases}.$$

Summing this over ℓ produces

$$6: \quad S \cdot \sum_{\ell=1}^H \ell = p \cdot \mathcal{W} + \left[\sum_{\ell \in \mathcal{P}} r_\ell \right] + p \cdot \mathcal{N} + \left[\sum_{\ell \in \mathcal{G}} r_\ell \right].$$

On $[1..H]$, recall that $\ell \mapsto r_\ell$ is a permutation up to sign. Thus

$$6': \quad \sum_{\ell=1}^H \ell = \left[\sum_{\ell \in \mathcal{P}} r_\ell \right] - \left[\sum_{\ell \in \mathcal{G}} r_\ell \right].$$

Subtracting equations, (6) – (6'), yields that

$$7: \quad [S-1] \cdot \sum_{\ell=1}^H \ell = p \mathcal{W} + p \mathcal{N} + 2 \cdot \sum_{\ell \in \mathcal{G}} r_\ell.$$

But S is odd, so $S-1 \equiv_2 0$. Reducing each side mod 2, then, gives

$$7': \quad \begin{aligned} 0 &\equiv_2 p \cdot [\mathcal{W} + \mathcal{N}] + 0 \\ &\equiv_2 \mathcal{W} + \mathcal{N}, \quad \text{since } p \text{ is odd.} \end{aligned}$$

Thus $\mathcal{W} \equiv_2 \mathcal{N}$, as desired. \blacklozenge

8: The Quadratic Reciprocity Theorem. For odd-primes p and q :

$$8': \quad \left(\frac{q}{p} \right) = \left(\frac{p}{q} \right) \cdot [-1]^{H_p \cdot H_q}.$$

When $p \neq q$ then we can write in a visually more symmetric way as

$$8'': \quad \left(\frac{q}{p} \right) \cdot \left(\frac{p}{q} \right) = [-1]^{H_p \cdot H_q}.$$

Equivalently, $\left(\frac{q}{p} \right)$ equals $\left(\frac{p}{q} \right)$ unless both p and q are 4NEG primes; in that case, the Legendre symbols are negatives of each other. \blacklozenge

Proof of (8''). Eisenstein has us decompose the rectangle in Figure 1 (page 3) into two triangles, in order to establish

$$9: \quad \mathcal{W}_p(q) + \mathcal{W}_q(p) = H_p \cdot H_q.$$

Then the Eisenstein Lemma shows that LhS(9) has the same parity as $\mathcal{N}_p(q) + \mathcal{N}_q(p)$, i.e.,

$$9': \quad \mathcal{N}_p(q) + \mathcal{N}_q(p) \equiv_2 H_p \cdot H_q.$$

Applying the Gauss Lemma now establishes (8''). \blacklozenge

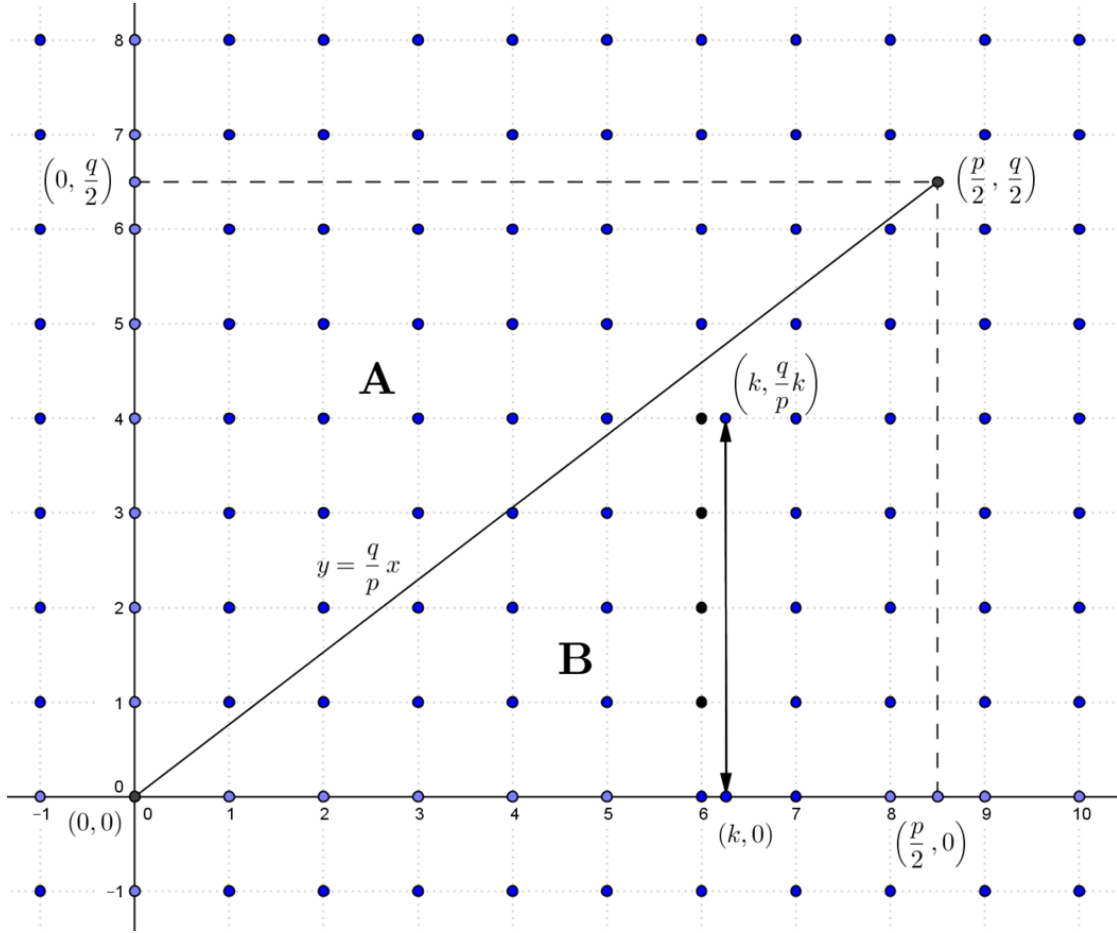


Fig.1: Here, $p=17$ and $q=13$. NOTE: The diagonal of $(0, H_p] \times (0, H_q]$ has no lattice-points, since $p \perp q$. Triangle **B** contains $\mathcal{W}_p(q)$ many lattice-pts because, traveling vertically from point $(k, 0)$, one passes through $\left\lfloor k \cdot \frac{q}{p} \right\rfloor$ many lattice-pts until reaching the diagonal line. Similarly, triangle **A** has $\mathcal{W}_q(p)$ lattice-points. Hence $\mathcal{W}_p(q) + \mathcal{W}_q(p) = H_p \cdot H_q$. (Image copied from Proof Wiki.)

Legendre/Jacobi Symbol

Consider a posint N and an integer $\boxed{k \perp N}$. This k is an N -**QR**, an N -quadratic-residue, if *there exists* an integer x with $\boxed{x^2 \equiv_N k}$; otherwise, this k is an N -**nonQR**. In contrast, if $k \not\perp N$, then k is *neither* a QR nor a nonQR.

For p prime, the **Legendre Symbol**

$$\left(\frac{k}{p}\right) := \begin{cases} 0 & \text{if } k \mid p \\ +1 & \text{if } k \text{ is a } p\text{-QR} \\ -1 & \text{if } k \text{ is a } p\text{-nonQR} \end{cases}.$$

I pronounce $\left(\frac{k}{p}\right)$ as “ k legendre p ”. Below, I use $\langle \cdot \rangle_N$ for the **symmetric** residue mod N .

10: Legendre-Symbol Thm (LSThm). For all odd-primes p and all integers K, k_ℓ, k'_ℓ , we have:

A: If $k \perp p$: Our k is a p -QR IFF $\left(\frac{k}{p}\right) = 1$. [By defn.]

$$\text{B: } \left(\frac{K}{p}\right) = \left\langle K^{\frac{p-1}{2}} \right\rangle_p.$$

Furthermore

i: LS is “top multiplicative”:

$$\left(\frac{k_1 \cdot k_2 \cdot \dots \cdot k_L}{p}\right) = \left(\frac{k_1}{p}\right) \cdot \left(\frac{k_2}{p}\right) \cdot \dots \cdot \left(\frac{k_L}{p}\right).$$

ii: If $k' \equiv_p k$ then $\left(\frac{k'}{p}\right) = \left(\frac{k}{p}\right)$. ◇

Pf of (B). Use the involution $x \mapsto \frac{K}{x}$ on $\Phi(p)$. Etc..♦

Pf of (i). Note $\text{RhS}(B)$ is [totally] multiplicative in K . Hence $\text{LhS}(B)$ is multiplicative in K . ♦

Defn of Jacobi Symbol. Factoring a **posodd** N into primes, $N = p_1 \cdot p_2 \cdots p_L$, define the **Jacobi Symbol** by

$$\left(\frac{k}{N}\right) := \left(\frac{k}{p_1}\right) \cdot \left(\frac{k}{p_2}\right) \cdots \left(\frac{k}{p_L}\right),$$

where k is an arbitrary integer. □

11: Commentary. Properties (13ii,iii,iv), below, will give a lightning-bolt (ie, Euclidean) algorithm for rapidly computing Jacobi-Symbols; the QRecip property of JS is the primary reason for generalizing LS. *However, something is lost* in the process:

For example, $\left(\frac{2}{9}\right) = 1$, yet certainly 2 is **not** a 9-QR, since 2 is not a 3-QR.

Also, $2^{H_9} = 2^4 \equiv_9 -2$. So the **symm-residue** $\langle 2^{H_9} \rangle_9$ **doesn't equal ± 1** , let alone answer whether 2 is a 9-QR. Similarly, $[-1]^{H_9} = 1$. So the value **is** in $\{\pm 1\}$, but the answer is *wrong*: Negative-one is a 9-nonQR, since -1 is a 3-nonQR. □

12: Prop'n. For odd integers d and e :

$$H_d + H_e \equiv_2 H_{de}. \quad \diamond$$

Proof. Write $d = 1 + 2A$ and $e = 1 + 2B$. The product de equals $4AB + 2A + 2B + 1$. Thus

$$H_{de} = 2AB + A + B \equiv_2 A + B \stackrel{\text{note}}{=} H_d + H_e. \quad \diamond$$

13: Jacobi-Symbol Thm (JSThm). For all posodd N, D, d_j , and all integers K, k_ℓ, k' :

A: For each $k \perp N$: k is an N -QR IFF

$$\text{Every prime } p \bullet N \text{ has } \left(\frac{k}{p}\right) = 1.$$

Moreover

i: JS is “multiplicative, top and bottom”:

$$\left(\frac{k_1 \cdot k_2 \cdots k_L}{N}\right) = \left(\frac{k_1}{N}\right) \cdot \left(\frac{k_2}{N}\right) \cdots \left(\frac{k_L}{N}\right) \quad \text{and}$$

$$\left(\frac{K}{d_1 \cdot d_2 \cdots d_J}\right) = \left(\frac{K}{d_1}\right) \cdot \left(\frac{K}{d_2}\right) \cdots \left(\frac{K}{d_J}\right).$$

ii: If $k' \equiv_N k$ then $\left(\frac{k'}{N}\right) = \left(\frac{k}{N}\right)$.

iii: These Jacobi-symbols satisfy:

$$\left(\frac{2}{N}\right) = \begin{cases} +1 & \text{if } N \equiv_8 \pm 1 \\ -1 & \text{if } N \equiv_8 \pm 3 \end{cases}.$$

$$\left(\frac{-1}{N}\right) = \begin{cases} +1 & \text{if } N \equiv_4 +1 \\ -1 & \text{if } N \equiv_4 -1 \end{cases}.$$

iv: QReciprocity: For n and d posodd,

$$\left(\frac{d}{n}\right) = \left(\frac{n}{d}\right) \cdot [-1]^{H_d \cdot H_n}. \quad \diamond$$

Pf of (13A). Fix a prime $p \bullet N$. Take r , a mod- p sqroot of k . Let $\mathbf{E} \in \mathbb{Z}_+$ be largest st. $p^{\mathbf{E}} \bullet N$. Use Hensel's lemma to lift r to s_p , a mod- $p^{\mathbf{E}}$ sqroot of k . [Details: Our r is a mod- p root of $f(x) := x^2 - k$. Now $f'(r) \stackrel{\text{note}}{=} 2r$ is not divisible by p , since p is odd. Thus Hensel's says this root can be lifted to a mod p^2 root, which can be lifted to a mod p^3 root, ..., indefinitely. ♥¹]

For each $p \bullet N$, let s_p be a mod- p sqroot of k . Use CRTm ♥² to suture together the $\{s_p \mid p \bullet N\}$ values into a mod- N sqroot of k . ♦

Pf (13iii). Let $\langle \cdot \rangle$ and \equiv mean **symm-residue mod 8**.

Our Two-is-QR Lemma implies that $\left(\frac{2}{N}\right) = -1$ IFF N has *oddly many* 8Far primes in its factorization.

OTOHand, $\langle N \rangle$ is the product of $\langle p \rangle$ over these primes. And for each two values in $\{\pm 3\}$, the product is congruent to ± 1 . So $\langle N \rangle = \pm 3$ IFF N has *oddly many* 8Far primes in its factorization. ♦

Pf (13iv). If $d \not\perp n$ then both $\left(\frac{d}{n}\right)$ and $\left(\frac{n}{d}\right)$ are zero. So establishing

$$\dagger: \quad \left(\frac{d}{n}\right) \cdot \left(\frac{n}{d}\right) \stackrel{?}{=} [-1]^{H_d \cdot H_n}$$

will suffice, since WLOG $d \perp n$.

Lets prove the following.

‡: Suppose each of d and e satisfies (†) w.r.t n . Then their product $d \cdot e$ satisfies (†) w.r.t n .

♥¹Fix $V \in [0 \dots p]$ with $V \equiv_p f'(r)$. For a posint ℓ , suppose r_ℓ is mod- p^ℓ sqroot of k . Compute the integer $m_\ell := -f(r_\ell)/p^\ell$. Now doing division mod p , compute $t_\ell \in [0 \dots p]$ st. $t_\ell \cdot V \equiv_p m_\ell$. Then $r_\ell + [t_\ell \cdot p^\ell]$ is mod- $p^{\ell+1}$ sqroot of k .

♥²The Chinese Remainder Theorem.

Applying (\dagger) twice, and mult. top and bottom,

$$\begin{aligned} \left(\frac{de}{n}\right) \cdot \left(\frac{n}{de}\right) &= \left(\frac{d}{n}\right) \left(\frac{e}{n}\right) \cdot \left(\frac{n}{d}\right) \left(\frac{n}{e}\right) \\ &= \left(\frac{d}{n}\right) \left(\frac{n}{d}\right) \cdot \left(\frac{e}{n}\right) \left(\frac{n}{e}\right) \\ &= [-1]^{H_d \cdot H_n} \cdot [-1]^{H_e \cdot H_n}. \end{aligned}$$

The combined exponent is $H_d H_n + H_e H_n$. ie.,

$$\left(\frac{de}{n}\right) \cdot \left(\frac{n}{de}\right) = [-1]^{[H_d + H_e] \cdot H_n}.$$

And Prop'n 12 says that the RhS equals $[-1]^{H_{de} \cdot H_n}$.

Inducting twice. W.r.t. a posodd N , say that posodd d is “ N -good” if

$$\pounds: \quad \left(\frac{d}{N}\right) = \left(\frac{N}{d}\right) \cdot [-1]^{H_d \cdot H_N}.$$

Having established (\dagger) , we have this:

\pounds : For each posodd N , the set of N -good numbers is sealed (closed) under multiplication.

Fixing a prime N , the QReciprocity Thm, in form (8'), tells us that every prime, d , is N -good. By (\pounds) , then: Every posodd d is N -good.

But (\pounds) is symmetric in N & d . So we can restate our accomplishment as: W.r.t. each posodd d , every prime N is d -good. Applying (\pounds) again, now says that every posodd N is d -good. \blacklozenge

1st Application of LST+QRecip. Fix an $N \in \mathbb{Z}$. We seek a characterization of those oddprimes $p \perp N$, for which $N \in \text{QR}_p$. Say k is **5Near** if $k \equiv_5 \pm 1$, and k is **5Far** if $k \equiv_5 \pm 2$.

14: Thm. Prime $p \neq 5$ has $\text{QR}_p \ni 5$ IFF p is 5Near. \blacklozenge

Proof. Since 5 is 4POS, we have $\left(\frac{5}{p}\right) = \left(\frac{p}{5}\right)$
Unfinished: as of 1Mar2023 \blacklozenge

2nd Application of LST+QRecip. We will show:

15a: For each $n \geq 2$, integer $[n^3 - 1]$ has a 3POS prime factor.

Note, $n^3 - 1 = [n - 1][n^2 + n + 1]$ where We will prove this stronger statement:

15b: Thm. Let $T_n := n^2 + n + 1$, and define

$$\mathcal{C}_n := \left\{ p \bullet T_n \mid p \text{ is prime, and } p \notin \{2, 3\} \right\}.$$

Every \mathcal{C}_n -prime is 3POS. And for $n \geq 2$, collection \mathcal{C}_n is not empty \blacklozenge

Pf of 3POSness. ISTShow that $p \in 3\text{POS}$, where p is an arbitrary non-2,3 prime p that divides

$$*: \quad F_n := 4T_n \stackrel{\text{note}}{=} [2n + 1]^2 + 3.$$

Now $F_n \equiv_p 0$, so $(*)$ says $[-3]$ is a mod- p square. By hyp, $-3 \perp p$, so $-3 \in \text{QR}_p$, i.e

$$\begin{aligned} 1 &= \left(\frac{-3}{p}\right) \stackrel{\text{LST}}{=} \left(\frac{-1}{p}\right) \cdot \left(\frac{3}{p}\right) \\ &\stackrel{\text{LST+QRecip}}{=} [-1]^{\frac{p-1}{2}} \cdot \left[[-1]^{\frac{p-1}{2}} \left(\frac{p}{3}\right)\right] = \left(\frac{p}{3}\right). \end{aligned}$$

But the *only* 3-QR is 1. So p is 3POS. \blacklozenge

Pf $\mathcal{C}_n \neq \emptyset$. Fix $n \geq 2$. FTSOC suppose \mathcal{C}_n is empty. Since T_n is odd, this implies that $T_n = 3^k$, for some $k \geq 2$; this last, since $T_n > T_1 = 3^1$. So $\boxed{F_n = 4 \cdot 3^k}$.

Since $F_n \equiv_3 0$, our $(*)$ says that $[2n + 1]^2 \mid 3$. Courtesy FTA, $[2n + 1] \mid 3$. Thus, $[2n + 1]^2 \equiv_9 0$.

Recall that $k \geq 2$, whence $F_n \equiv_9 0$. So $(*)$ implies that $0 \equiv_9 3$, which is false. Hence \mathcal{C}_n is non-void. \blacklozenge

E11: For what *negative* integers n do we have (15a)? Or have (15b)?

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