

Pythagorean Triples

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Entrance. Let “ $n \equiv_4 k$ ” mean^{♥1} $4 \nmid [n - k]$.

A **Pythagorean triple** $\langle a, b, c \rangle$ of integers satisfies

$$P0: \quad a^2 + b^2 = c^2.$$

It is a **PPT**, a **primitive Pythagorean triple**, if, in addition,

$$P1: \quad a, b, c \in \mathbb{Z}_+.$$

$$P2: \quad \text{GCD}(a, b, c) = 1. \quad \left[\begin{array}{l} \text{Necessarily, each pair} \\ \text{of } a, b, c \text{ is coprime,} \\ \text{courtesy (P0).} \end{array} \right]$$

$$P3: \quad a \text{ is odd. [We'll see that } b \text{ is even, and } c \text{ is odd.]}$$

By (P2), a and b can't *both* be even; so there is no loss of generality in the (P3) normalization. Could also b be odd? If yes, then $a^2 + b^2 \equiv_4 1 + 1 = 2$. But c is even, so $c^2 \equiv_4 0$. ✗.

Here is one correspondence, and its inverse:

$$1a: \quad \begin{array}{l} c = \frac{1}{2}[y^2 + x^2]. \\ b = \frac{1}{2}[y^2 - x^2]. \\ a = yx. \end{array} \quad \begin{array}{l} \sqrt{c+b} = y. \\ \sqrt{c-b} = x. \end{array}$$

A reworking gives this corr., and its inverse:

$$1b: \quad \begin{array}{l} c = r^2 + q^2. \\ b = 2rq. \\ a = r^2 - q^2. \end{array} \quad \begin{array}{l} \sqrt{\frac{1}{2}[c+a]} = r. \\ \sqrt{\frac{1}{2}[c-a]} = q. \end{array}$$

To make this precise, say that a pair (x, y) is an **odd-pair** if:

$$D1: \quad x, y \in \mathbb{Z}_+. \text{ Additionally, } 1 \leq x < y.$$

$$D2: \quad x \perp y.$$

$$D3: \quad \text{Both } x \text{ and } y \text{ are odd.}$$

^{♥1}Use \equiv_N to mean “congruent mod N ”. Let $n \perp k$ mean that n and k are co-prime [no prime in common].

Use $k \nmid n$ for “ k divides n ”. Its negation $k \nmid n$ means “ k does not divide n .” Use $n \nmid k$ and $n \nmid k$ for “ n is/is-not a multiple of k .” Finally, for p a prime and E a natnum: Use double-verticals, $p^E \nmid n$, to mean that E is the **highest** power of p which divides n . Or write $n \nmid p^E$ to emphasize that this is an assertion about n . Use **PoT** for Power of Two and **PoP** for Power of (a) Prime.

A pair (r, q) is a **mixed-pair** if:

$$M1: \quad r, q \in \mathbb{Z}_+. \text{ Furthermore, } r > q \geq 1.$$

$$M2: \quad r \perp q.$$

$$M3: \quad \text{Integers } q \text{ and } r \text{ have opposite parities.}$$

Given $x, y \in \mathbb{Z}$, define

$$2a: \quad \mathbf{T}(x, y) := \langle a, b, c \rangle, \quad \text{where } \begin{array}{l} a := yx; \\ b := \frac{1}{2}[y^2 - x^2]; \\ c := \frac{1}{2}[y^2 + x^2]. \end{array}$$

$$2b: \quad \mathbf{U}(r, q) := \langle a, b, c \rangle, \quad \text{where } \begin{array}{l} a := r^2 - q^2; \\ b := 2rq; \\ c := r^2 + q^2. \end{array}$$

3a: Odd-pair Thm. *There is a 1-to-1 correspondence between odd-pairs and primitive triples: The map*

$$\text{Odd-Pairs} \xrightarrow{\mathbf{T}} \text{Primitive Triples}$$

is a bijection. ♦

Our \mathbf{T} is well-defined. Necessarily, $a \stackrel{\text{def}}{=} yx$ is odd and a, b and c are positive since $1 \leq x < y$. Since x^2 and y^2 are each odd, and $\text{ODD} \pm \text{ODD}$ is even, we have that b and c are indeed integers.

To establish that $\langle a, b, c \rangle$ is primitive, what remains is to prove that $\text{GCD}(a, b, c) = 1$. So suppose that p is a prime dividing a . Necessarily, p divides x or y ; WLOGenerality $p \nmid x$. Were p to divide b , forcing $p \nmid [y^2 - x^2]$, then $p \nmid y^2$ and consequently $p \nmid y$. But that contradicts that (x, y) is an odd-pair. Thus $a \perp b$. ♦

$\mathbf{T}(\cdot)$ is injective. Suppose another odd-pair (w, z) also gives rise to the same triple

$$\mathbf{T}(w, z) = \langle a, b, c \rangle = \mathbf{T}(x, y).$$

Since $wz = a = xy$, without loss of generality $w < x$ and $z > y$. But then

$$b = \frac{1}{2}[z^2 - w^2] > \frac{1}{2}[y^2 - x^2] = b.$$

This is a contradiction. ♦

$\mathbf{T}(\cdot)$ is surjective. Fix a primitive triple $\langle a, b, c \rangle$. My first goal is to produce integers x & y so that $\mathbf{T}(x, y) = \langle a, b, c \rangle$. To this end, define X & Y by

$$Y := c + b, \quad X := c - b. \quad \text{Thus,}$$

$$Y \cdot X = [c + b] \cdot [c - b] \stackrel{\text{note}}{=} c^2 - b^2 = a^2.$$

Necessarily, $Y, X \in \mathbb{Z}_+$. I want to let

$$\forall: \quad x := \sqrt{X} \quad \text{and} \quad y := \sqrt{Y},$$

so I need to show that X & Y are squares. Since product YX is a square, ISTShow that $X \perp Y$. Fixing a prime $p \nmid X$, then, ISEstablish that

p does *not* divide Y .

Were p to also divide Y then it would divide the two linear combinations

$$Y + X \stackrel{\text{note}}{=} 2c \quad \text{and} \quad Y - X \stackrel{\text{note}}{=} 2b.$$

But $p \nmid X \nmid a^2$, so p is odd. Thus $p \nmid c$ and $p \nmid b$, contradicting (P1).

Final step. I've shown that y & x from (\forall) are posints. Since the argument also showed that $Y \perp X$, we now have $y \perp x$. And (P1, P0, P3) show that $c > b > 0$; so $Y > X$ and thus $y > x \geq 1$. Lastly, y & x are each odd, since each divides a , which is odd. \blacklozenge

3b: Mixed-pair Thm. *This map is a bijection:*

$$\text{Mixed-Pairs} \xrightarrow{\mathbf{U}} \text{Primitive Triples}. \quad \blacklozenge$$

Pf. Define $f((y, x)) := (r, q)$, where $r := \frac{1}{2}[y + x]$ and $q := \frac{1}{2}[y - x]$. Reversing, $g((r, q)) := (y, x)$, where $y := r + q$ and $x := r - q$. Easily, f and g are well-defined on the rationals, and are inverses of each other. So the proof will be finished when you show (exercise) that $f(\text{odd-pair}) \in \text{Mixed-pair}$ and $g(\text{mixed-pair}) \in \text{Odd-pair}$. \blacklozenge

Special case of Fermat's Last Theorem

Fermat proved the following theorem.

4: FLT for $N = 4$. *There is no posint soln to either of these:*

$$\begin{aligned} \dagger: & \quad a^4 + b^4 = c^4 ; \\ \ddagger: & \quad D^4 + E^4 = U^2 . \end{aligned} \quad \diamond$$

Prelim. A (\dagger) -triple yields a (\ddagger) -triple $a^4 + b^4 = [c^2]^2$, hence ISTShow: \nexists soln to (\ddagger) . Below, define \perp so that $x \perp y$ means: $x \perp y$ and $x \not\equiv_2 y$.

Let expression " $\alpha \in \square$ " mean that there exists a posint β with $\alpha = \beta^2$. ♦

Proof. FT SOC, suppose we have a (\ddagger) which has minimum U , over all (\ddagger) . Were there a prime with $p \bullet D$ and $p \bullet E$, then $p^2 \bullet U$, so $(\frac{D}{p}, \frac{E}{p}, \frac{U}{p^2})$ is a smaller (\ddagger) ; ✗. Thus $D \perp E$, so (D^2, E^2, U) is a PPT (prim. Pythag. triple), since $[D^2]^2 + [E^2]^2 = U^2$; WLOG D is odd and E even.

Our PPT parametrization (1b) yields posints $R > Q$ with $R \perp Q$ such that

$$\begin{aligned} *1: & \quad U = R^2 + Q^2 ; \\ *2: & \quad E^2 = 2RQ ; \\ & \quad D^2 = R^2 - Q^2 . \end{aligned}$$

The latter is $D^2 + Q^2 = R^2$. As $R \perp Q$, this last is a PPT. Since D odd, nec. Q is even and R is odd. Hence $R \perp 2Q$, so $(*2)$ implies that

$$\pounds: \quad R \in \square \quad \text{and} \quad 2Q \in \square .$$

PPT $D^2 + Q^2 = R^2$ engenders posints $r > q$ with $r \perp q$, such that

$$\begin{aligned} *3: & \quad R = r^2 + q^2 ; \\ *4: & \quad Q = 2rq ; \\ & \quad D = r^2 - q^2 . \end{aligned}$$

Our (\pounds) implies \exists posint c with $2Q = [2c]^2$, i.e. $2c^2 = Q = 2rq$, by $(*4)$. Thus $c^2 = rq$. But $r \perp q$,

so $r, q \in \square$. This, together with (\pounds) , tells us that there exist posints u, d, e such that

$$R = u^2, \quad r = d^2 \quad \text{and} \quad q = e^2 .$$

Consequently, we can restate $(*3)$ as

$$**3: \quad d^4 + e^4 = u^2 .$$

This has form (\ddagger) . Moreover, $u \leq u^2 = R < U$ courtesy $(*1)$. We see to our relief that $(**3)$ contradicts the minimality of U . ♦

Pell's Equation

What are all the Pythagorean triples

$$5: \quad a^2 + [a+1]^2 = c^2,$$

i.e., where the legs of the triangle are consecutive integers? I'll show, that if (a, c) is a soln, then so is

$$6: \quad \begin{aligned} a' &:= 3a + 2c + 1 & \text{and} \\ c' &:= 4a + 3c + 2. \end{aligned}$$

The first nine values are

a	c	a	c	a	c
0	1	119	169	23660	33461
3	5	696	985	137903	195025
20	29	4059	5741	803760	1136689

The set of soln-pairs has a group structure, with multiplication $\begin{bmatrix} a_1 \\ c_1 \end{bmatrix} \odot \begin{bmatrix} a_2 \\ c_2 \end{bmatrix}$ described by this formula:

$$7: \quad \begin{aligned} a' &:= -a_2 + c_2 - a_1 - 2a_1a_2 + 2a_1c_2 \\ &\quad + c_1 + 2c_1a_2 - c_1c_2 - 1; \\ c' &:= -c_1 - 2c_1a_2 + 2c_1c_2 \\ &\quad + 2a_2 - c_2 + 2a_1 + 4a_1a_2 - 2a_1c_2 + 1. \end{aligned}$$

The group-inverse of an elt: $\begin{bmatrix} a' \\ c' \end{bmatrix} := \begin{bmatrix} a \\ c \end{bmatrix}^{\odot -1}$ where

$$7_{\text{inv}}: \quad \begin{aligned} a' &:= -3a + 2c - 2; \\ c' &:= -4a + 3c - 2. \end{aligned}$$

E.g., $\begin{bmatrix} 3 \\ 5 \end{bmatrix}^{\odot -1} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Eqn (5) is equivalent to $[2a]^2 + [2a+2]^2 = [2c]^2$. We can write this in form $f(a, c) = 0$, but the f polynomial is not homogeneous. We can get a homogeneous eqn: Setting

$$d := 2a + 1$$

makes the eqn $[d-1]^2 + [d+1]^2 = [2c]^2$. Dividing by 2 and rewriting gives

$$8: \quad d^2 - 2c^2 = -1.$$

This is special case of what the “*generalized Pell's eqn* with *Pell coefficient* μ ”:

$$\varepsilon_{\mu}^{\tau}: \quad d^2 - \mu c^2 = \tau, \quad \text{where } \mu \in \mathbb{Z}_+.$$

We'll ask: What *targets* $\tau \in \mathbb{Z}$ admit a soln pair (d, c) in *integers*? And: When τ admits a soln, what is the *complete* set of soln-pairs?

Just “Pell's eqn” shall mean the $\tau=1$ case:

$$\varepsilon_{\mu}: \quad d^2 - \mu c^2 = 1.$$

This always has the two *trivial solns* $(\pm 1, 0)$.

We will find all *consecutive pythag-triples* by solving (8), which we will get from the soln set of (ε_2) .

Group structure

Below, the objects $\mathcal{G}, \mathcal{P}, \mathcal{M}, \dots$ all depend on μ . When needed, I'll indicate the dependence as \mathcal{G}_{μ} , with a subscript.

Use $\mathbf{I} := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{J} := \begin{bmatrix} 0 & \mu \\ 1 & 0 \end{bmatrix}$. For the generic *lin-comb* (linear combination) of matrices, use

$$\mathbf{M} := \mathbf{M}_{d,c} := d\mathbf{I} + c\mathbf{J} = \begin{bmatrix} d & \mu c \\ c & d \end{bmatrix} \quad (\text{always with letters “d” and “c”}).$$

The set

$$\mathcal{L} := \left\{ \mathbf{M}_{d,c} \mid d, c \in \mathbb{Z} \right\}$$

of lin-combs is sealed under multiplication, since \mathbb{Z} is a ring and $\mathbf{J}^2 = \mu\mathbf{I} \in \mathcal{L}$. Thus

9: $(\mathcal{L}, \cdot, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix})$ is a commutative semi-group.

Note $\text{Det}(\mathbf{M}) = d^2 - \mu c^2$. The following set is a group

$$\mathcal{G} := \left\{ \mathbf{M} \in \mathcal{L} \mid \text{Det}(\mathbf{M}) \in \{1, -1\} \right\},$$

because \mathcal{G} is sealed under mult., since $\{1, -1\}$ is. And \mathcal{G} is sealed under inverse, since

$$10: \quad \begin{aligned} \mathbf{M}^{-1} &= \frac{1}{\text{Det}(\mathbf{M})} \begin{bmatrix} d & -\mu c \\ -c & d \end{bmatrix} \stackrel{\text{note}}{=} \hat{d}\mathbf{I} + \hat{c}\mathbf{J}, \text{ where} \\ \hat{d} &:= \frac{d}{\text{Det}(\mathbf{M})} \quad \text{and} \quad \hat{c} := \frac{-c}{\text{Det}(\mathbf{M})}. \end{aligned}$$

Note \hat{d} and \hat{c} are integers, since $\text{Det}(\mathbf{M}) = \pm 1$.

Define two sets by the condition above them

$$\mathcal{G} = \begin{matrix} \text{Det}(\mathbf{M})=+1 \\ \mathcal{P} \end{matrix} \sqcup \begin{matrix} \text{Det}(\mathbf{M})=-1 \\ \mathcal{N} \end{matrix}.$$

The map $\text{Det} : \mathcal{G} \rightarrow \{+1, -1\}$ is a gp-homomorphism, so

$$\mathcal{P} := \text{Det}^{-1}(\{1\}).$$

is a subgroup of \mathcal{G} .

11: Lemma. *For those values of μ with \mathcal{N}_μ not empty, then \mathcal{N}_μ is a \mathcal{G} -coset of \mathcal{P}_μ .* \diamond

Proof. Fix a matrix $\mathbf{N}_\mu \in \mathcal{N}_\mu$. For each $\mathbf{M} \in \mathcal{N}_\mu$, note that $\mathbf{M} \cdot \mathbf{N}_\mu \stackrel{\text{Det}}{=} [-1]^2 = 1$. I.e, $\mathbf{MN}_\mu \in \mathcal{P}$. \blacklozenge

Remark. When $\mu = 2$: Matrix $\mathbf{N}_2 := \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ shows that \mathcal{N}_2 is not empty, hence is (the only other coset) of \mathcal{P}_2 .

In contrast, \mathcal{N}_3 is empty. Indeed, *lots* of μ have $\mathcal{N}_\mu = \emptyset$; see the **Appendix**. \square

Henceforth μ is a posint

Upper-lefthand entry. Define two sets

$$\mathcal{P} = \begin{matrix} d > 0 \\ \mathcal{U} \end{matrix} \sqcup \begin{matrix} d < 0 \\ \mathcal{U}^- \end{matrix};$$

this \mathcal{U} is the set of $\begin{bmatrix} d & \mu c \\ c & d \end{bmatrix}$ with $d > 0$.

12: Lem. $[\mu \in \mathbb{Z}_+] (\mathcal{U}, \cdot, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix})$ is a group. \diamond

Proof. When $\mathbf{M} \in \mathcal{U}$, necessarily $\mathbf{M}^{-1} \in \mathcal{U}$, by (10).

Now consider a product of two matrices in \mathcal{U} :

$$\begin{bmatrix} \alpha & \mu\beta \\ \beta & \alpha \end{bmatrix} \cdot \begin{bmatrix} d & \mu c \\ c & d \end{bmatrix} = \begin{bmatrix} \alpha d + \mu\beta c & ? \\ \alpha c + \beta d & ? \end{bmatrix};$$

here $d, \alpha \in \mathbb{Z}_+$ and $c, \beta \in \mathbb{Z}$ and

$$\begin{aligned} *1: \quad \alpha^2 &= 1 + \mu\beta^2 > \mu\beta^2; \\ d^2 &= 1 + \mu c^2 > \mu c^2. \end{aligned}$$

Now $\mu \geq 0$, so the RhSes are non-negative; hence

$$*2: \quad |\alpha d|^2 > |\mu\beta c|^2.$$

To show \mathcal{U} sealed under product we need to prove that $\alpha d + \mu\beta c \stackrel{?}{>} 0$. I.e, that $\alpha d \stackrel{?}{>} -\mu\beta c$. So $|\alpha d| \stackrel{?}{>} |\mu\beta c|$ certainly suffices, since αd is positive. And this is implied by (*2). \blacklozenge

When \mathcal{U} non-trivial. Suppose that \mathcal{U}_μ is not the trivial gp $\{\mathbf{I}\}$. So we can pick the *minimal* posint Υ for which there exists a $\Delta \in \mathbb{Z}_+$ with $\Delta^2 - \mu\Upsilon^2 = 1$. This Δ is unique. W.r.t Δ or Υ ,

$$\mathbf{S} = \mathbf{S}_\mu := \begin{bmatrix} \Delta & \mu\Upsilon \\ \Upsilon & \Delta \end{bmatrix}$$

is the minimal positive elt of \mathcal{U} .

As an example, $\mathbf{S}_2 = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$.

13: Lem. *Fix an $\mathbf{M} \in \mathcal{U}$. Then the map $h_{\mathbf{M}} : \mathbb{Z} \rightarrow \mathbb{Z}$ by*

$$h_{\mathbf{M}}(n) := c\text{-coord}(\mathbf{S}^n \cdot \mathbf{M})$$

is strictly increasing. \diamond

Pf. Our goal is $c' < c$, where

$$\begin{aligned} \begin{bmatrix} d' & \mu c' \\ c' & d' \end{bmatrix} &:= \mathbf{S}^{-1} \cdot \mathbf{M} = \begin{bmatrix} \Delta & -\mu\Upsilon \\ -\Upsilon & \Delta \end{bmatrix} \cdot \begin{bmatrix} d & \mu c \\ c & d \end{bmatrix} \\ &= \begin{bmatrix} \Delta c - \Upsilon d & ? \\ \Delta c - \Upsilon d & ? \end{bmatrix}. \end{aligned}$$

So our goal is i.e $\Delta c - \Upsilon d < c$, i.e $[\Delta - 1]c < \Upsilon d$. WLOG $c > 0$ (since $\Delta \in \mathbb{Z}_+$ so $\Delta - 1 \geq 0$), and so our goal becomes $\frac{\Delta - 1}{\Upsilon} < \frac{d}{c}$. Its LhS ≥ 0 , so establishing

$$*: \quad \left[\frac{\Delta - 1}{\Upsilon} \right]^2 < \left[\frac{d}{c} \right]^2$$

suffices. But $d^2 = \mu c^2 + 1$, so $\text{RhS}(*) > \mu$. Hence showing $\left[\frac{\Delta - 1}{\Upsilon} \right]^2 \leq \mu$ suffices, i.e $[\Delta - 1]^2 \leq \mu\Upsilon^2$, i.e $\Delta^2 - 2\Delta + 2 \leq 1 + \mu\Upsilon^2$ suffices. But this is equivalent to $-2\Delta + 2 \leq 0$. This latter is true since $\Delta \in \mathbb{Z}_+$. \blacklozenge

14: Prop'n. *Imagine an $\mathbf{M} := \begin{bmatrix} d & \mu c \\ c & d \end{bmatrix} \in \mathcal{U}$. Suppose*

$$15: \quad c \geq 0 > c',$$

where $\begin{bmatrix} d' & \mu c' \\ c' & d' \end{bmatrix} := \mathbf{S}^{-1} \cdot \mathbf{M}$. Then $\mathbf{M} = \mathbf{I}$. \diamond

Proof. Whoa! 03Nov2009: This proof was empty. \blacklozenge

16: Theorem. $[\mu \in \mathbb{Z}_+]$ *The map $f : \mathbb{Z} \rightarrow \mathcal{U}$ given by $f(n) := \mathbf{S}^n$, is a bijection.* \diamond

Appendix

Which values of μ have \mathcal{N}_μ non-empty? Although we won't give a complete answer here, we will rule out many. Say that an integer D is **4NEG** if $D \equiv_4 -1$; and it is **4POS** if $D \equiv_4 +1$.

Consider our eqn $d^2 - \mu c^2 = -1$, but modulo a prime $p \nmid \mu$. This gives

$$17: \quad d^2 \equiv_p -1.$$

The Legendre-symbol thm implies that (17) has a soln, d , IFF p is 4POS or is 2.

18: Lemma. *Consider a posint μ . Then congruence*

$$19: \quad d^2 \equiv_\mu -1$$

has a soln IFF our μ is a product of powers of 4POS-primes and, possibly, one copy of the prime 2. \diamond

Proof. A 4POS prime p admits a soln to (17). Now Hensel's lemma applies^{♥2} to give us a solution modulo an arbitrary power, p^n .

Certainly $d^2 \equiv -1$ has a soln modulo 2 (but modulo 4, it certainly does *not*). Now the Chinese Remainder Thm allows us to stitch these congruence-solns together, \blacklozenge

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^{♥2}It is the easy non-singular case, since p is odd.