There is one order-complete ordered-field

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§Contents

Rings. In a ring $(\Gamma, +, \mathbf{0}_\Gamma, \cdot, \mathbf{1}_\Gamma)$, if there is a posint n so that $\mathbf{1}_\Gamma + \mathbf{1}_\Gamma + \overset{n}{\cdots} + \mathbf{1}_\Gamma$ equals $\mathbf{0}_\Gamma$, then the *smallest* such n is called the ***characteristic*** of the ring, and I write $\text{Char}(\Gamma) = n$. If no such posint exists, then I will write $\text{Char}(\Gamma) = \infty$; however, the *standard term* is $\text{Char}(\Gamma) = 0$, and you will see this in algebra texts and in some of my notes.

A ring is ***commutative*** (abbrev., ***comm-ring***) if its multiplication is commutative. In a comm-ring Γ, a ***zero-divisor*** $\alpha \in \Gamma$ admits a non-zero elt $\beta \in \Gamma$ (this β need not be unique) so that $\alpha\beta = \mathbf{0}_\Gamma$. Use **ZD** for *zero-divisor*. [Letting $\equiv$ denote $\equiv_{12}$, in the $\mathbb{Z}_{12}$ ring, 9 is a ZD, since $9 \cdot 8 \equiv 0$, yet $8 \not\equiv 0$. OTOHand, even though $5 \cdot 24 \equiv 0$, this doesn't show that 5 is a $\mathbb{Z}_{12}$–ZD, since $24 \equiv 0$.]

An ***integral domain*** Γ is a *commutative* ring with no ZDs except for $\mathbf{0}_\Gamma$, the ***trivial* ZD**. If the characteristic of an *integral domain* is finite, then $\text{Char}(\Gamma)$ is a prime number. In particular, this holds if Γ is a field.

1: Fact. *If Γ is a field of finite **order** (finite cardinality) then $|\Gamma| = p^k$ for some prime p and posint k. Conversely, for each such prime p and $k \in \mathbb{Z}_+$, there exists a field of order p^k, and this field is unique up to field-isomorphism.* ◊

Partial proof. For the prime $p := \text{Char}(\Gamma)$, there is a copy of $\mathbb{Z}_p$ inside Γ, making Γ a $\mathbb{Z}_p$-vectorspace. Letting k be the dimension of this vectorspace, then, we obtain $|\Gamma| = p^k$.

The remaining Facts take a fair amount of work to prove. ♦

Totally-Ordered Sets. A ***TOS*** $(\Gamma, \prec)$ has an antireflexive, transitive relation $\prec$ so that for each $\alpha \neq \beta$ in Γ, either $\alpha \prec \beta$ or $\alpha \succ \beta$.

A subset $S \subset \Gamma$ is "***order-dense*** in Γ" if:

> *For each pair $\alpha \prec \beta$ of elements in Γ, there exists $\tau \in S$ with $\alpha \prec \tau \prec \beta$.*

If S is order-dense as a subset of *itself*, then say that "S is ***order self-dense***". [E.g, TOS $(\mathbb{Q}, <)$ is order self-dense, but $(\mathbb{Z}, <)$ is not.]

***Least upper-bound property* [LUBP].** In TOS $(\Gamma, \prec)$, consider sets $A, B \subset \Gamma$ and a point $\gamma \in \Gamma$. Let

2.1:
$$A \preccurlyeq \gamma \quad \text{mean} \quad [\forall \alpha \in A, \text{ necessarily } \alpha \preccurlyeq \gamma]\,;$$
$$A \preccurlyeq B \quad \text{mean} \quad [\forall \alpha \in A \text{ and } \forall \beta \in B\text{: } \alpha \preccurlyeq \beta]\,.$$

An ***upper-bound*** for a set $A \subset \Gamma$ is an element $\gamma \in \Gamma$ such that $A \preccurlyeq \gamma$. Use $\mathsf{UB}_\Gamma(A)$ for the *set* of upper-bnds, and $\mathsf{LB}_\Gamma(A)$ for the lower-bnd–set. (Dispense with the subscript if clear from context.) Our $(\Gamma, \prec)$ has the ***LUBP*** if:

2.2: *Each non-void $A \subset \Gamma$ which is upper-bnded [i.e $\mathsf{UB}_\Gamma(A) \neq \varnothing$] has a least upper-bound. That is, $\mathsf{UB}_\Gamma(A)$ has a minimum element.*

Reversing the inequalities yields the ***greatest lower-bound property***, abbreviated ***GLBP***.

The LUB of a set A (when it *has* a LUB!) is called the ***supremum*** of the set, and is written $\sup(A)$ or $\sup_\Gamma(A)$. Similarly, the ***infimum*** is the GLB, written $\inf(A)$.

2.3: LUBP theorem. *TOS $(\Gamma, \prec)$ has the LUBP IFF it has the GLBP.* ◊

Proof of [LUBP ⇒ GLBP]. Fix a non-void lower-bnded subset $B \subset \Gamma$; so $A := \mathsf{LB}_\Gamma(B)$ is non-empty. My goal is to produce a (hence the) greatest lower-bound for B, using that

†: $$A \stackrel{\text{def}}{=} \mathsf{LB}_\Gamma(B)\,, \quad \text{and}$$
‡: $$\mathsf{UB}_\Gamma(A) \supset B\,.$$

Since $\mathsf{UB}_\Gamma(A) \supset B \neq \varnothing$, and A is non-void, the LUBP applies, and tells us that $\lambda := \sup_\Gamma(A)$ exists. In particular

$$\dagger': \qquad \lambda \succcurlyeq A\,.$$

Since λ is the *least* upper-bnd, $\lambda \preccurlyeq \mathsf{UB}_\Gamma(A) \supset B$ and so $\lambda \preccurlyeq B$. Restating, λ *is* a lower-bound of B. (Note: λ *might* or *might not* be in B.)

And, by (†) and (†′), this λ dominates each lower-bound of B. So λ is a *greatest* lower-bound of B. ♦

Making a Real Assumption. A TOS $(\Gamma, \prec)$ satisfing LUBP [equivalently, GLBP] is said to be ***order-complete***. We take as an axiom [or derive via *Dedekind cuts* or *Cauchy sequences*] that

2.4: $(\mathbb{R}, <)$ *is order-complete.*

This means that the extended reals, $\overline{\mathbb{R}}$, satisfies a slightly stronger property: *Each*[♡1] *subset* $A \subset \overline{\mathbb{R}}$ *has a* $\sup(A)$ *and an* $\inf(A)$ *in* $\overline{\mathbb{R}}$. In consequence, sup() and inf() are maps from the full $\mathcal{P}(\overline{\mathbb{R}}) \to \overline{\mathbb{R}}$.

Ordered-fields

An ***ordered-field*** $(\Gamma, +, \mathbf{0}, \cdot, \mathbf{1}; <)$ is a field which is an ordered set satisfying $\forall \alpha,\beta,\tau \in \Gamma$:

i: If $\alpha < \beta$ then $\alpha+\tau < \beta+\tau$. That is, relation "<" is **translation invariant**.

ii: If $\alpha,\beta > \mathbf{0}$ then their product $\alpha\cdot\beta > \mathbf{0}$. I.e: "*Product is Positivity-Preserving*".

3: Ordered-field lemma. *In an ordered-field* Γ*:*

a: *If* $\alpha \neq \mathbf{0}$*, then* $[\alpha > \mathbf{0}] \Leftrightarrow [-\alpha < \mathbf{0}]$.

b: *Fix* $\alpha > \beta$*. If* $\mu > \mathbf{0}$ *then* $\mu\alpha > \mu\beta$*. If* $\mu < \mathbf{0}$ *then* $\mu\alpha < \mu\beta$*. Also, if* $\mu \geq \mathbf{0}$ *then* $[\alpha \geq \beta] \Rightarrow [\mu\alpha \geq \mu\beta]$*. If* $\mu \leq \mathbf{0}$ *then* $[\alpha \geq \beta] \Rightarrow [\mu\alpha \leq \mu\beta]$.

Now suppose $\alpha_j > \beta_j > \mathbf{0}$*, for* $j \in \{1,2\}$*. Then* $\alpha_1\alpha_2 > \beta_1\beta_2 > \mathbf{0}$.

c: *For each* $\alpha \neq \mathbf{0}$*, necessarily* $\alpha^2 > \mathbf{0}$*. Hence* $\mathbf{1} > \mathbf{0}$.

d: $\mathrm{Char}(\Gamma) = \infty$.

e: *If* $\mathbf{0} < \alpha < \beta$*, then* $\mathbf{0} < 1/\beta < 1/\alpha$. ◊

Note to self: All but (e) holds for an *ordered integral-domain*. Adapt this material for that generalization, then do the field-of-quotients construction.]

Proof of (a). By translation invariance, if $\alpha > \mathbf{0}$ then $\alpha - \alpha > \mathbf{0} - \alpha$; etc.

Pf of (b). Saying $\alpha > \beta$ means $\alpha - \beta > \mathbf{0}$, hence (ii) implies $\mu[\alpha - \beta] > \mathbf{0}$, etc. Or if μ is negative, then $[-\mu][\alpha - \beta] > \mathbf{0}$; so now (a) and associativity of mult (and that $-[-\mu] = \mu$) together imply that $\mu[\alpha - \beta] < \mathbf{0}$.

The two versions with *non*-strict inequality, "≥", follow from the strict inequalities.

Lastly, suppose $\alpha_j > \beta_j > \mathbf{0}$. Multiplying the 2[nd] by β_1 gives $\beta_1\alpha_2 > \beta_1\beta_2$. Multiplying the 1[st] by α_2 gives $\alpha_1\alpha_2 > \beta_1\alpha_2$. Transitivity yields $\alpha_1\alpha_2 > \beta_1\beta_2$. ♦

Pf of (c). If $\mathbf{1}$ negative, then (b) implies that $\mathbf{1}\cdot\mathbf{1}$ is positive; a contraction. And $\mathbf{1} \neq \mathbf{0}$, by the axioms for a field. Thus *trichotomy* forces that $\mathbf{1}$ is positive. ♦

Pf of (d). [I temporarily rename $\mathbf{0}$ to $\mathbf{0}_\Gamma$ and $\mathbf{1}$ to $\mathbf{1}_\Gamma$.] By (b), we now know that $\mathbf{0}_\Gamma < \mathbf{1}_\Gamma$. Since "<" is translation-invariant, induction implies that for each posint n, the sum $\mathbf{1}_\Gamma + \mathbf{1}_\Gamma + \overset{n}{\ldots} + \mathbf{1}_\Gamma$ is positive. ♦

Pf of (e). For $\gamma > \mathbf{0}$, its mult-inverse $1/\gamma$ is not $\mathbf{0}$ (since *it* has a mult-inv). Were $1/\gamma < \mathbf{0}$, then (b) implies $\gamma \cdot \frac{1}{\gamma} \overset{\text{note}}{=\!=} \mathbf{1}$ is negative, contradicting (c). Hence

$$\gamma > \mathbf{0} \quad \text{implies} \quad 1/\gamma > \mathbf{0}\,.$$

By (ii), product $\alpha\beta$ is positive, so $\frac{1}{\alpha\beta} > \mathbf{0}$. Multiplying the given $\mathbf{0} < \alpha < \beta$ by $\frac{1}{\alpha\beta}$ yields, courtesy (b), that $\mathbf{0} < 1/\beta < 1/\alpha$. ♦

[♡1] E.g, $\sup(\varnothing) = -\infty$ and $\inf(\varnothing) = +\infty$. Indeed, for $A \subset \overline{\mathbb{R}}$: $[A \neq \varnothing] \iff [\inf(A) \preccurlyeq \sup(A)]$.

4: *Defn.* For $n \in \mathbb{N}$ and $\beta \in \Gamma$, denote $\beta + \overset{n}{\cdots} + \beta$. by $n\beta$. For $n \in \mathbb{Z}_-$, use $n\beta$ for $[-\beta] + \overset{-n}{\cdots} + [-\beta]$. Use $\mathbb{Z}_\Gamma$ for $\{n{\cdot}\mathbf{1}_\Gamma \mid n \in \mathbb{Z}\}$, etc. □

5: *Exer. E1.* Fix a field $(\Gamma, +, \mathbf{0}, \cdot, \mathbf{1})$. For $n \in \mathbb{Z}$ let $\widehat{n}$ denote $n{\cdot}\mathbf{1}$ in Γ, as defined in (4).

i: For integers n_j and $d_j \in \mathbb{Z}$ with each $\widehat{d_j} \neq \mathbf{0}$, prove: *If* $\frac{n_1}{d_1} = \frac{n_2}{d_2}$ *then* $\widehat{n_1}/\widehat{d_1} = \widehat{n_2}/\widehat{d_2}$.

ii: Suppose $p := \mathrm{Char}(\Gamma)$ is finite, hence prime. Prove $n \mapsto \widehat{n}$ is a *ring*-homomorphism of $\mathbb{Z}$ into Γ. Show that $n \mapsto \widehat{n}$ can be interpreted as an *injective* field-homomorphism of $\mathbb{Z}_p \hookrightarrow \Gamma$.

[For two fields $\mathbf{F}$ and $\mathbf{G}$, is it *possible* to have a non-injective field-hom $\mathbf{F} \to \mathbf{G}$?]

iii: Now suppose $\mathrm{Char}(\Gamma) = \infty$. Argue that for each $q \in \mathbb{Q}$, value $\widehat{q} \in \Gamma$ is *well-defined* by choosing integers $d \neq 0$ and n st. $q = \frac{n}{d}$, and then defining $\widehat{q} := \widehat{n}/\widehat{d}$.

iv: Suppose $\mathrm{Char}(\Gamma) = \infty$, Prove that $q \overset{f}{\mapsto} \widehat{q}$ is an injective field-homomorphism of $\mathbb{Q} \hookrightarrow \Gamma$. Further, if $(\Gamma, \prec)$ is an *ordered* field, then f is *order-preserving* as a map $(\mathbb{Q}, <) \hookrightarrow (\Gamma, \prec)$. □

Archimedean fields

An ordered-field Γ is ***Archimedean*** if for each $\tau \in \Gamma$ there exists a natnum n with $n\mathbf{1}_\Gamma \geq \tau$. By setting $k := n+1$, we see this is equivalent to: $\exists k \in \mathbb{N}$ *with* $k\mathbf{1}_\Gamma > \tau$. Equivalently, $\mathsf{UB}_\Gamma(\mathbb{Z}_\Gamma)$ is empty.

6.1: Archy lemma. *If ordered-field $(\Gamma, +, \mathbf{0}_\Gamma, \cdot, \mathbf{1}_\Gamma; <)$ is Archimedean, then for each $\beta \succ \mathbf{0}_\Gamma$: The upper-bound set of $M_\beta := \{n\beta \mid n \in \mathbb{N}\}$ is empty.*

Moreover, if there exists a $\beta \in \Gamma$ with $\mathsf{UB}_\Gamma(M_\beta)$ empty, then Γ is Archimedean. ♢

Pf. Fix a posint K with $K\mathbf{1}_\Gamma \geq \mathbf{1}_\Gamma/\beta \overset{\text{note}}{>} \mathbf{0}$. Multiply by β to conclude that $K\beta \geq \mathbf{1}_\Gamma$, by (3). So for each natnum n, element $[nK] \cdot \beta$ dominates $n\mathbf{1}_\Gamma$. Thus $\mathsf{UB}_\Gamma(M_\beta) \subset \mathsf{UB}_\Gamma(\mathbb{Z}_\Gamma)$. And $\mathsf{UB}_\Gamma(\mathbb{Z}_\Gamma) = \varnothing$.

Exercise E2: The converse is left the Reader. ♦

6.2: Corollary. *Suppose ordered-field $(\Gamma, +, \mathbf{0}_\Gamma, \cdot, \mathbf{1}_\Gamma; <)$ is Archimedean. Then $\mathsf{UB}_\Gamma(\mathbb{Z}_\Gamma) = \varnothing = \mathsf{LB}_\Gamma(\mathbb{Z}_\Gamma)$. Moreover, for each $\alpha \in \Gamma$, there exists a unique integer K with $[K-1]\mathbf{1}_\Gamma \leq \alpha < K\mathbf{1}_\Gamma$.* ♢

Proof. The order-reversing map $x \mapsto -x$ on Γ sends lower-bnd-sets to upper-bnd-sets, etc., hence $\mathsf{LB}(\mathbb{Z}_\Gamma)$ is empty.

Setting $U_k := \{x \in \Gamma \mid x \geq k\mathbf{1}_\Gamma\}$, the foregoing tell us that $\bigcup_{k \in \mathbb{Z}} U_k$ is Γ, and $\bigcap_{k \in \mathbb{Z}} U_k$ is empty. These sets are *nested*, $\ldots, U_{-1} \supset U_0 \supset U_1 \supset U_2 \supset \ldots$ so there is a unique integer $K \in \mathbb{Z}$ with the given α in the difference-set $U_{K-1} \smallsetminus U_K$. ♦

7: OC ⇒ Archimedean theorem. *Suppose ordered-field $(\Gamma, +, \mathbf{0}_\Gamma, \cdot, \mathbf{1}_\Gamma; <)$ is order-complete. Then Γ is Archimedean.* ♢

Proof. FTSOContradiction, suppose $\mathsf{UB}(\mathbb{Z}_\Gamma)$ is not empty; so $\mathbb{Z}_\Gamma$ has a Γ–LUB; call it τ. Now $\tau - \mathbf{1}_\Gamma$ is less than τ, hence cannot upper-bnd $\mathbb{Z}_\Gamma$. Consequently, $\exists n \in \mathbb{Z}_+$ with $[n-1]\mathbf{1}_\Gamma > \tau - \mathbf{1}_\Gamma$. Thus $n\mathbf{1}_\Gamma > \tau$, which is a blatant contradiction. ♦

8.0: Order-dense lemma. *Fix $(\Gamma, +, \widehat{0}, \cdot, \widehat{1}; <)$, an Archimedean field. For $q \in \mathbb{Q}$, define $\widehat{q} \in \Gamma$ as in (5), and let $\widehat{\mathbb{Q}}$ denote the copy of $\mathbb{Q}$ inside Γ. Then $\widehat{\mathbb{Q}}$ is order-dense in Γ.* ♢

Proof. Fix $\alpha < \beta$ in Γ. We will produce integers $N \in \mathbb{Z}_+$ and K such that

$$*: \qquad \alpha < \widehat{K/N} < \beta\,.$$

By (6.1) there is posint N with $N[\beta - \alpha] > \widehat{2}$. Dropping the "$\widehat{\ }$" symbol for the rest of the proof, we have $\frac{\beta-\alpha}{2} > \frac{1}{N}$. Hence $\alpha + \frac{1}{N} < \alpha + \frac{\beta-\alpha}{2}$, so

$$**: \qquad \alpha + \tfrac{1}{N} < \beta\,.$$

Courtesy (6.2), $\exists K \in \mathbb{Z}$ with $[K-1] \leq N\alpha < K$, i.e, $N\alpha < K \leq [N\alpha] + 1$. Hence $\alpha < \frac{K}{N} \leq \alpha + \frac{1}{N}$, since N is positive. This and (**), yield (*). ♦

Complete ordered-field(s)

We now come to the main result.

9: Theorem. *Suppose* $(\Gamma, +, \widehat{0}, \cdot, \widehat{1}; <)$ *and* $(\mathbf{F}, +, \overline{0}, \cdot, \overline{1}; <)$ *are ordered-fields. Then they are ordered-field–isomorphic. Moreover, there is a unique OF-isomorphism between them.* ♢

Proof (sketch). For $S \subset \mathbb{Q}$, let $\widehat{S} := \{\widehat{q} \mid q \in S\} \overset{\text{note}}{\subset} \Gamma$. Define similarly $\overline{S} \subset \mathbf{F}$. For $\alpha \in \Gamma$ and $x \in \mathbf{F}$, define

$$U_\alpha := \{q \in \mathbb{Q} \mid \widehat{q} \le \alpha\} \quad \textit{and} \quad V_x := \{q \in \mathbb{Q} \mid \overline{q} \le x\}.$$

There exist $q, r \in \mathbb{Q}$ with $\alpha - \widehat{1} < q < \alpha < r < \alpha + \widehat{1}$; this, by (8.0), density. Hence U_α is non-void and upper-bnded in $\mathbb{Q}$, so $\overline{U_\alpha}$ is non-void and upper-bnded in $\mathbf{F}$. I.e, $\sup_{\mathbf{F}}(\overline{U_\alpha})$ exists in $\mathbf{F}$. Consequently, we have a well-defined map $\Phi{:}\Gamma \to \mathbf{F}$, by

$$\Phi(\alpha) := \sup_{\mathbf{F}}(\overline{U_\alpha}).$$

Evidently Φ is *weakly* order-preserving in that for all $\alpha,\beta \in \Gamma$: $[\alpha \le \beta] \Rightarrow [\Phi(\alpha) \le \Phi(\beta)]$.

Similarly, $\mathcal{G}(x) := \sup_\Gamma(\widehat{V_x})$ is a weakly-OP map $\mathcal{G}{:}\mathbf{F} \to \Gamma$. ♦

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