Base-$\mathcal{V}$ numerals

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Note: I intend to eventually precede this with a discussion of base-$\mathcal{V}$ representations of just *integers*.

Numerals. Fix a ***base*** $\mathcal{V} \in [2 \,..\, \infty)$. A ***vigit*** [base-$\mathcal{V}$ digit] is an element of $J_{\mathcal{V}} := [0 \,..\, \mathcal{V})$. A ***base-$\mathcal{V}$ numeral*** is an infinite string, $\boldsymbol{\sigma}$, of form

$$\dagger: \quad \boldsymbol{\sigma} := d_K \cdots d_1 d_0 \bullet d_{-1} d_{-2} d_{-3} \cdots, \text{ with each } d_j \in J_{\mathcal{V}}$$

and $K \in [-1 \,..\, \infty)$. We call the "$\bullet$" the ***vig-point***. When $\mathcal{V}$=10 we may also call "$\bullet$" the ***decimal point***. [Note: $K = -1$ means there are no vigits before the point.] We regard two numerals as the ***same numeral***, if we can alter one into the other by adjoining/removing leading zero-vigits. I.e,

$$7.2945 \cdots \quad \text{and} \quad 007.2945 \cdots$$

are regarded as the same numeral.

An overbar indicates to ∞ly repeat a finite string of vigits. E.g,

$$34.2\overline{581}$$

abbreviates the $34.2\,581\,581\,581 \cdots$ numeral. The ***eventual-period*** of this string is 3, since string "581" has 3 vigits. When $\mathcal{V}$:=ten, period=1 and the repetition string is "0", e.g, $2.74\overline{0}$, then some folks write this as 2.74 and call it a **"**terminating decimal**"**, though other conventions for the missing digits are possible.

Numbers. The number represented by $\boldsymbol{\sigma}$ in (†) is

$$\ddagger: \qquad \mathrm{Num}_{\mathcal{V}}(\boldsymbol{\sigma}) \quad := \sum_{j \in (-\infty \,..\, K]} d_j \cdot \mathcal{V}^j \,.$$

For example, in base $\mathcal{V}$:=ten, the numeral $0.7\overline{9}$ represents $\frac{7}{10} + \sum_{k=2}^{\infty} [9/10^k]$, i.e $\frac{7}{10} + \frac{1}{10}$, that is, $\frac{4}{5}$. Another base-ten representation for $\frac{4}{5}$ is the $0.8\overline{0}$ numeral. Numerals

$$0.7\overline{9} \quad \text{and} \quad 0.8\overline{0} \quad (\text{in base-ten})$$

are decidedly **<u>not</u>** the same *numeral*, but they do <u>name</u> the same *number*.

*1: *Base-twelve example.* Using *0,1,...,9,A,B* for base-twelve vigits, these two base-twelve numerals

$$.04\overline{B} \quad \text{and} \quad .05\overline{0}$$

are names for the number 5/144. □

The ***$\mathcal{V}$-adic rationals*** are those that can be written in form $N/\mathcal{V}^k$, with $N \in \mathbb{Z}$ and $k \in \mathbb{N}$.

*2: Multiple-rep Thm. *Fix $\mathcal{V}$. A posreal has multiple base-$\mathcal{V}$ numerals IFF it is a $\mathcal{V}$-adic rational.*

Moreover, a (positive) $\mathcal{V}$-adic rational x has precisely two base-$\mathcal{V}$ reps. Write $y := \mathcal{V}^S x$ for the unique integer S which places y in interval $[\frac{1}{\mathcal{V}}, 1)$. Use $\mathsf{H} := \mathcal{V} - 1$ for the largest vigit. Then one rep for y looks like

$$R_{\mathsf{H}}: \qquad \bullet d_1 d_2 \cdots d_{L-1}\, f\, \overline{\mathsf{H}}\,, \quad \text{with } L \in \mathbb{N},$$

where each $d_j \in [0 \,..\, \mathsf{H}]$ and $f \in [0 \,..\, \mathsf{H})$. Numeral

$$R_0: \qquad \bullet d_1 d_2 \cdots d_{L-1}\, [f{+}1]\, \overline{0}$$

is the other base-$\mathcal{V}$ representation. *Pf.* Exercise E1.◊

Fraction→eventually-periodic. Consider $x := \frac{N}{D}$, where $N, D \in \mathbb{Z}_+$. Write each posint in base-$\mathcal{V}$,

$$N = f_L \cdots f_1 f_0 \quad \text{and} \quad D = d_K \cdots d_1 d_0 \,,$$

then divide numeral $d_K \cdots d_0$ into $f_L \cdots f_0$, using the long-division algorithm. You will obtain[♡1] an eventually-periodic base-$\mathcal{V}$ rep of x. Moreover (exercise E2), its eventual-period is less than D.

*3: *Example: Eventually-periodic→fraction.* We'll write $x := 3.511\overline{27}$ (base-ten) as a fraction.

Let $y := 10^3 x$, to move the periodic part to the vig-point. The periodic part has period P=2. So

$$10^2 \cdot y = 351127.\overline{27} \quad \text{and}$$
$$y = 3511.\overline{27}\,.$$

Subtracting yields that

$$[10^2 - 1]y = 351127 - 3511 \overset{\text{note}}{=\!=} 347616\,.$$

[♡1] When x is a $\mathcal{V}$-adic rational, you will obtain the (R_0) rep. To obtain the (R_{H}) rep: At stage L in the long-division, substract one less copy of D than you would have. Stage $L{+}1$ will have you dividing D into $\mathcal{V} \cdot D$. Subtract only H copies of D and proceed to stage $L{+}2$. Continue.

Solving for y gives $y = \frac{347616}{99} \overset{\text{note}}{=\!=} \frac{38624}{11}$. Hence

$$x = \frac{y}{10^3} = \frac{38624}{11 \cdot 1000} = \frac{4828}{1375},$$

as was sought. □

General: Eventually-periodic→fraction. For your eventually-periodic base-$\mathcal{V}$ numeral for x, take $S \in \mathbb{Z}$ so that $y := \mathcal{V}^S \cdot x$ has numeral

$$y = d_K \cdots d_1 d_0 \bullet \overline{f_1 \cdots f_P}.$$

Subtacting the bottom from the top, in

$$\begin{aligned} \mathcal{V}^P \cdot y &= d_K \cdots d_1 d_0 f_1 \cdots f_P \bullet \overline{f_1 \cdots f_P} \quad \text{and} \\ y &= \qquad\quad d_K \cdots d_1 d_0 \bullet \overline{f_1 \cdots f_P}, \end{aligned}$$

gives a posint N which is this difference

$$[d_K \cdots d_0 f_1 \cdots f_P] - [d_K \cdots d_0]$$

of base-$\mathcal{V}$ numerals. Thus $[\mathcal{V}^P - 1]y = N$, so

*4: $$x = \frac{N}{[\mathcal{V}^P - 1] \cdot \mathcal{V}^S}.$$

Finally, put RhS(**??**) in standard form, so that the numerator and denominator are integers with no common factor.

*5: *Base-twelve computation.* With $\mathcal{V} := 12$ and using *0*,*1*,...,*9*,*A*,*B* for twigits, our goal is to write

$$x := \bullet 5\overline{B2}$$

as a ratio of posints (each written in, say, base-ten).

From above, $S = 1$ and $P = 2$. Moreover, N equals

$$5B2 - 5 \overset{\text{note}}{=\!=} [5 \cdot 12^2 + 11 \cdot 12 + 2] - 5 = 849.$$

Thus x equals

$$\frac{849}{[12^2 - 1] \cdot 12^1} = \frac{849}{143 \cdot 12} = \frac{283}{143 \cdot 4} = \frac{283}{572},$$

as desired.

If we wish x written as a ratio of base-twelve numerals, we could convert $\frac{283}{572}$. Alternatively, simply do the above computations in base-twelve. E.g, $5B2 - 5 = 5A9$. And

$$[\mathcal{V}^P - 1] \cdot \mathcal{V}^S = [100 - 1] \cdot 10 = BB \cdot 10 = BB0.$$

Thus $x = \frac{5A9}{BB0}$. Dividing out *3* from numer & denom,

$$x = \frac{1B7}{3B8},$$

with numerator coprime to denominator. □

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