Nullspaces of commuting transformations : LinearAlg

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23 February, 2022 (at *01:38*)

ABSTRACT: Conditions under which the nullspace of a composition $\mathsf{B}\circ\mathsf{A}$ is the span of the two nullspaces. This is then applied to constant-coefficient linear differential equations.

§A Entrance

The setting is a vectorspace $\mathbf{V}$ and two linear transformations $\mathsf{A},\mathsf{B}:\mathbf{V}\circlearrowleft$. Use both $\mathrm{Nul}(\mathsf{A})$ and A° for the nullspace of transformation A. Use C for the composition $\mathsf{C} := \mathsf{BA}$.

1: Fact. *Suppose* $\mathsf{B},\mathsf{A}:\mathbf{V}\to\mathbf{V}$. *Then*

$$2:\quad \mathrm{Dim}(\mathsf{B}^\circ)+\mathrm{Dim}(\mathsf{A}^\circ) \overset{1':}{\geq} \mathrm{Dim}\Big(\mathrm{Nul}(\mathsf{BA})\Big) \geq \mathrm{Dim}(\mathsf{A}^\circ)\,. \quad \Diamond$$

Now suppose that $\mathsf{A} \leftrightarrows \mathsf{B}$ (the trns commute) so that $\mathsf{C} = \mathsf{BA} = \mathsf{AB}$. Then, automatically,

$$3:\quad \mathsf{C}^\circ \supset \mathrm{Spn}(\mathsf{B}^\circ, \mathsf{A}^\circ)\,.$$

We explore when we have equality in (1′), and when in (3). For two subspaces $\mathbf{W},\mathbf{W}' \subset \mathbf{V}$, let $\mathbf{W} \perp \mathbf{W}'$ mean that $\{\mathbf{W},\mathbf{W}'\}$ is a (*linearly*) *independent set*, as in (∗), below. [In particular, $\mathbf{W}$ and $\mathbf{W}'$ only intersect in the singleton $\{\mathbf{0}\}$.] More generally, $\perp_{j=1}^{K}\mathbf{W}_j$ indicates mutual independence of the subspaces in that the *only* solution to

$$*:\quad \mathbf{w}_1 + \cdots + \mathbf{w}_K = \mathbf{0} \text{ with each } \mathbf{w}_j \in \mathbf{W}_j,$$

is to have every $\mathbf{w}_j$ be $\mathbf{0}$.

4: Commuting Thm. *Suppose that* $\mathsf{A} \leftrightarrows \mathsf{B}$. *If* $\mathsf{B}^\circ \perp \mathsf{A}^\circ$ *and at least one of the nullspaces is finite-dim'al, then*

$$3':\quad \mathsf{C}^\circ = \mathrm{Spn}(\mathsf{B}^\circ, \mathsf{A}^\circ)\,. \quad \Diamond$$

5: Corollary. *Consider* $\mathsf{C} := {\mathsf{B}_1}^{R_1}{\mathsf{B}_2}^{R_2}\cdots{\mathsf{B}_K}^{R_K}$ *where* $R_j \in \mathbb{N}$, *and the* $\mathsf{B}_1,\dots,\mathsf{B}_K$ *are commuting trns with at most one* B_j *having an* ∞*-dim'al nullspace. If* $\perp_{j=1}^{K}\mathrm{Nul}({\mathsf{B}_j}^{R_j})$ *then*

$$4':\quad \mathsf{C}^\circ = \mathrm{Spn}\Big(\mathrm{Nul}({\mathsf{B}_1}^{R_1}),\dots,\mathrm{Nul}({\mathsf{B}_K}^{R_K})\Big)\,.$$

Further suppose that each B_j *has a* <u>*1-dim'al*</u> *nullspace. If each* $\mathrm{Nul}({\mathsf{B}_j}^{R_j})$ *has dimension at least* R_j (*and thus has dimension exactly* R_j, *by* (1′)) *then* $\mathrm{Dim}(\mathsf{C}^\circ)$ *is precisely* $R_1+\cdots+R_K$. $\Diamond$

CEX: C1. We do not get equality in (1′) even if the trns commute and $\mathbf{V}$ is finite-dim'al: Let B be the idempotent matrix $\left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right]$ and $\mathsf{A} := \mathsf{B}$.

Moreover, equality (1′) *and* commutativity <u>and</u> $\mathrm{Dim}(\mathbf{V}) < \infty$ are not enough for (3′): Let B be the nilpotent matrix $\left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right]$. Then $\mathsf{A} := \mathsf{B}$ and B commute, yet $\mathrm{Nul}(\mathsf{B}^2)$ is 2-dim'al.

In both examples, *the nullspaces of* B *and* A *are* <u>***not***</u> *linearly independent.* □

CEX: C2. Here is an example of $\mathsf{A} \leftrightarrows \mathsf{B}$ with linearly-indep nullspaces, yet inclusion (3) is strict. (Necessarily, this $\mathbf{V}$ is ∞-dim'al):

Let $\mathcal{E} := \{\mathbf{w},\mathbf{a}_0,\mathbf{a}_1,\mathbf{a}_2\dots,\mathbf{b}_0,\mathbf{b}_1,\mathbf{b}_2,\dots\}$ be a basis for $\mathbf{V}$. Define trn B to map

$$\dagger:\quad \mathbf{w} \mapsto \mathbf{a}_0 \text{ and each } \mathbf{a}_j \mapsto \mathbf{a}_{j+1},$$
$$\ddagger:\quad \text{and each } \mathbf{b}_j \mapsto \mathbf{0}\,.$$

Restricted to $\mathbf{W} := \mathrm{Spn}(\mathbf{w},\mathbf{a}_0,\dots)$, trn B has no nullspace, courtesy (†). Since B maps $\mathbf{W}\circlearrowleft$, we have equality

$$\mathsf{B}^\circ = \mathrm{Spn}(\mathbf{b}_0,\mathbf{b}_1,\dots)\,.$$

Define trn A on basis $\mathcal{E}$ using (†,‡) analogously, reversing the roles of vectors $\mathbf{a}_?$ and $\mathbf{b}_?$; thus A° is $\mathrm{Spn}(\mathbf{a}_0,\mathbf{a}_1,\dots)$. So $\mathsf{A}^\circ \perp \mathsf{B}^\circ$. And easily BA and AB are each the zero operator. Yet $\mathbf{w}$ is not in $\mathrm{Spn}(\mathsf{B}^\circ,\mathsf{A}^\circ)$. □

Proof of Commuting Thm, (4). Since $\mathsf{A} \leftrightarrows \mathsf{B}$, we can WLOG shrink $\mathbf{V}$ to be C°; so our goal is to show that *every* vector is in $\operatorname{Spn}(\mathsf{B}^\circ, \mathsf{A}^\circ)$. WLOG A° is finite dim'al. So fixing a vector $\mathbf{w} \in \mathbf{V}$, ISTFind an $\mathbf{a} \in \mathsf{A}^\circ$ so that

6: *Trn* B *sends* $\mathbf{w} - \mathbf{a}$ *to* $\mathbf{0}$.

Trn B maps all of $\mathbf{V}$ into A° (by commutativity). Thus $\mathsf{B}\lfloor_{\mathsf{A}^\circ}$ maps $\mathsf{A}^\circ\circlearrowleft$. But $\operatorname{Dim}(\mathsf{A}^\circ)$ is finite, so $\mathsf{B}\lfloor_{\mathsf{A}^\circ}$ maps A° *onto* A°. (Otherwise $\mathsf{B}\lfloor_{\mathsf{A}^\circ}$ would have to fail to be 1-to-1, ie, it would have nullspace. But the nullspaces of B and A are *linearly indep*, by hypothesis.)

The upshot is that $\mathsf{B}(\mathbf{w})$ is in A°, and there is a vector $\mathbf{a} \in \mathsf{A}^\circ$ which is also mapped to $\mathsf{B}(\mathbf{w})$ by B. Hence (6). ♦

§B Application to Differential Eqns

Let D and I be the differentiation and identity operators. That the nullspace of $\mathsf{D} - \mathrm{I}$ is $\operatorname{Spn}(\mathsf{e}^x)$ follows easily from the Mean Value Thm; in particular, this nullspace is *one-dimensional*. (Here, we are using expression e^x to mean the fnc $[x \mapsto \mathsf{e}^x]$.) More generally,

7: $$x^j \mathsf{e}^x \xmapsto{\mathsf{D}-\mathrm{I}} j \cdot x^{j-1}\mathsf{e}^x, \quad \text{for } j \in \mathbb{N}.$$

Courtesy (1), the dimension of $\operatorname{Nul}\big([\mathsf{D}-\mathrm{I}]^7\big)$ is at most 7. A small effort shows that the set of fncs $\{\mathsf{e}^x, x\mathsf{e}^x, x^2\mathsf{e}^x, \ldots, x^6\mathsf{e}^x\}$ is lin-indep, and so its span is precisely $\operatorname{Nul}\big([\mathsf{D}-\mathrm{I}]^7\big)$.

A linear substitution now shows that

8: $$\operatorname{Nul}([\mathsf{D}-9\mathrm{I}]^7) = \operatorname{Spn}(\mathsf{e}^{9x}, x\mathsf{e}^{9x}, x^2\mathsf{e}^{9x}, \ldots, x^6\mathsf{e}^{9x}).$$

Here "9" represents an arbitrary complex number, and "7" represents an arbitrary posint.

Consider an arbitrary complex monic polynomial $p(\mathsf{D}) = \mathsf{D}^N + b_{N-1}\mathsf{D}^{N-1} + \cdots + b_0\mathrm{I}$. Factor it as

$$p(\mathsf{D}) = [\mathsf{D} - z_1\mathrm{I}]^{R_1} \cdot \ldots \cdot [\mathsf{D} - z_K\mathrm{I}]^{R_K}$$

with distinct zeros $z_1, \ldots, z_K \in \mathbb{C}$. These K operators $[\mathsf{D} - z_j\mathrm{I}]^{R_j}$ commute and satisfy the remaining hypotheses of (5). Hence Corollary 5 applies to show that the set of fncs f satisfying this DE,

9: $$f^{(N)} + b_{N-1}f^{(N-1)} + \cdots + b_1 f' + b_0 f = 0,$$

is the span of

9′: $$\begin{array}{l} \mathsf{e}^{z_1 x},\ x\mathsf{e}^{z_1 x},\ x^2\mathsf{e}^{z_1 x},\ \ldots,\ x^{R_1-1}\mathsf{e}^{z_1 x}, \\ \mathsf{e}^{z_2 x},\ x\mathsf{e}^{z_2 x},\ x^2\mathsf{e}^{z_2 x},\ \ldots,\ x^{R_2-1}\mathsf{e}^{z_2 x}, \\ \vdots \\ \mathsf{e}^{z_K x},\ x\mathsf{e}^{z_K x},\ x^2\mathsf{e}^{z_K x},\ \ldots,\ x^{R_K-1}\mathsf{e}^{z_K x}. \end{array}$$

And this is the standard result from a diffyq course. In particular, the soln-set to (9) has dimension N.

Filename: `Problems/Algebra/LinearAlg/nullspace.latex`
As of: *Friday 08Aug2003.* Typeset: *23Feb2022 at 01:38.*