Mean Value Theorem and L'Hôpital's Rule

Jonathan L.F. King
University of Florida, Gainesville FL 32611-2082, USA
`squash@ufl.edu`
Webpage http://squash.1gainesville.com/
29 August, 2021 (at *17:17*)

Abbrevs. IVT; *Intermediate Value Thm.* MVT; *Mean Value Thm.* FTC; *Fund. Thm of Calculus.* Use cts for "continuous" and cty for "continuity".

The exponential fnc exp can also be written $\exp(x) = \mathsf{e}^x$. (So $\log \circ \exp = Id_{\mathbb{R}}$ and $\exp \circ \log = Id_{\mathbb{R}_+}$.) Use *NevZ* to mean "never-zero"; e.g "exp() is *NevZ* on $\mathbb{R}$".

Prolegomena. Use $\overline{\mathbb{R}}$ for the **extended reals** $[-\infty, +\infty]$. With DNE denoting "Does Not Exist", adjoin a point to $\overline{\mathbb{R}}$ to create

$$\mathbb{R}^{\circledast} := [-\infty, +\infty] \sqcup \{\mathsf{DNE}\}.$$

Use **diff'able** for "differentiable". A fnc $f{:}\mathbb{R}{\to}\mathbb{R}$ is **ext-diff'able** (for **extended diff'able**) at point 6 if the $\lim_{x \to 6} \frac{f(x)-f(6)}{x-6}$ exists in $[-\infty, +\infty]$. Use f'_{LH} for the **lefthand ext-derivative**, from $\lim_{x \nearrow 6}$. Use f'_{RH} for the **righthand ext-derivative**, $\lim_{x \searrow 6}$.

The result below apply to fncs on a closed bounded interval J. For specificity, I will use $J := [4, 6]$ and will use $J^{\circ} := (4, 6)$ for its interior.

A fnc $h{:}J{\to}\mathbb{R}$ has a (global) **max-point** $P \in J$ if

$$\forall x \in J: \qquad h(P) \geq h(x);$$

and the number $h(P)$ is the "**max-value** of h on J". Weaker, P is a **_local_ max-point** of h (on J) if there exists a J-open set $U \ni P$, so that P is a global max-point of $h{\downarrow}_U$. Imagine analogous defns for **min-point**, **min-value** and **local min-point** of h.

1: Tool. *A continuous $h{:}J{\to}\mathbb{R}$ has a max-point and a min-point.* **Proof.** Interval J is compact, etc. ◇

2: Lemma. *Suppose continuous $h{:}J{\to}\mathbb{R}$ has a local-extremum at a point $\tau \in J^{\circ}$. If h is extended-differentiable at τ, then* $\boxed{h'(\tau) = 0}$. ◇

Proof. WLOG τ is a local-min of h. So for all $x > \tau$ with x suff. close to τ, necessarily $h(x) - h(\tau) \geq 0$; thus $h'_{RH}(\tau) \in [0, +\infty]$. Similarly, $h'_{LH}(\tau) \in [-\infty, 0]$. By hypothesis, $h'_{LH}(\tau) = h'_{RH}(\tau)$. So $h'(\tau)$ is zero. ♦

3: Rolle's Thm. *Suppose a continuous $h{:}[4, 6]{\to}\mathbb{R}$ is ext-diff'able on $(4, 6)$. If $h(4) = h(6)$, then there exists a point $\tau \in (4, 6)$ such that* $\boxed{h'(\tau) = 0}$. ◇

Pf. Courtesy (2), WLOG $h()$ has no global-max in J°, so its global-max on J must be the common value of $h(4) = h(6)$. Thus h has a global-*min* in J°; pick one, and call it τ. Now (2) gives that $h'(\tau) = 0$. ♦

4: MVThm. *Suppose cts $f{:}[4, 6]{\to}\mathbb{R}$ is ext-diff'able on $(4, 6)$. Then there exists a $\tau \in (4, 6)$ such that*

$$f'(\tau) = \frac{f(6)-f(4)}{6-4}.$$ ◇

Pf. With $S := \frac{f(6)-f(4)}{6-4}$, the slope of the chord, let

$$h(x) := f(x) - [x-4] \cdot S.$$

Since $h(4) = f(4) = h(6)$, Rolle's thm applies to assert a point $\tau \in (4, 6)$ with $0 = h'(\tau) \overset{\text{note}}{=\!=\!=} f'(\tau) - 1 \cdot S$. ♦

E.g: Fnc $f(x) := \sqrt[3]{x}$ has $f'(0) = +\infty$. MVT applies to assert a point $\tau \in (-8, 27)$ where $f'(\tau)$ equals $\frac{3--2}{27--8}$.

For a beautiful MVT-application due to Liouville, see *Liouville's Theorem* on my TeachingPage. □

5: Prop'n. *On an interval $H \subset \mathbb{R}$, suppose fnc g is extended-differentiable. If g' is NevZ on H, then g is strictly-monotone on H. (We do **_not_** assume that g' is continuous.)* ◇

Proof. Were g not strictly-monotone then, WLOG, there are points $a < b < c$, in H, with $g(a) \leq g(b)$ yet $g(b) \geq g(c)$. WLOG $g(a) \leq g(c)$. Applying IVT to g on $[a, b]$ yields point $z \in [a, b]$ with $g(z) = g(c)$. Now Rolle's thm produces a $\tau \in (z, c)$ with $g'(\tau) = 0$. ♦

6: Cauchy MVT. *Continuous fncs $f, g{:}[4, 6]{\to}\mathbb{R}$ are diff'able on $(4, 6)$. Then there exists a $\tau \in (4, 6)$ st.*

6a: $f'(\tau) \cdot [g(6) - g(4)] = g'(\tau) \cdot [f(6) - f(4)].$

And if g' is never-zero on $(4, 6)$, then

6b: $$\frac{f'(\tau)}{g'(\tau)} = \frac{f(6) - f(4)}{g(6) - g(4)}.$$ ◇

Proof of (6a). Apply Rolle's theorem to

7:     $h(x) := f(x) \cdot [g(6) - g(4)] - g(x) \cdot [f(6) - f(4)]$,

noting that $h(4) = f(4)g(6) - f(6)g(4) = h(6)$.   ♦

Proof of (6b).   Since g' is *NevZ* on $(4,6)$, the MVT forces $g(6) \neq g(4)$. So we can cross-divide in (6a). ♦

L'Hôpital's rule

I'll usually state the thms for a one-sided limits, $\lim_{x \searrow T}$ or $\lim_{x \nearrow T}$, where $T \in \overline{\mathbb{R}}$. I'll use "$\mathrm{Lim}(f)$" to mean $\lim_{x \searrow T} f(x)$ or $\lim_{x \nearrow T} f(x)$, as appropriate.

First some *cautionary tales :*

8: *No end.*   What is $\lim_{x \searrow -\infty} \frac{\sqrt{x^2+1}}{x}$?   Well, $x<0$ implies that $-\sqrt{x^2} = x$, so

$$\frac{\sqrt{x^2+1}}{x} \quad = \quad \frac{\sqrt{x^2+1}}{-\sqrt{x^2}} \quad = \quad -\sqrt{1 + \frac{1}{x^2}} \;.$$

Hence the $\lim_{x \searrow -\infty}$ is –1. But applying L'H to $\frac{\sqrt{x^2+1}}{x}$ will (after algebra) give $\frac{x}{\sqrt{x^2+1}}$. Which L'H sends to the original $\frac{\sqrt{x^2+1}}{x}$.   □

9: *BttDB: Back to the Drawing Board.* Certainly

$$\lim_{x \searrow -\infty} \frac{x^2 + \sin(x)}{x^2} \quad = \quad 1 \,.$$

L'H examines $\frac{2x + \cos(x)}{2x}$; still lim=1. But applying L'H again gives $\frac{2 - \sin(x)}{2}$, which has *no limit*. There is no error here; if the f'/g' limit doesn't exist in $\overline{\mathbb{R}}$, then we can draw **no conclusion** about the f/g limit. □

10: L'Hôpital's Thm. *Fix a "target" $T \in [-\infty, +\infty)$ and consider real-valued fncs f, g defined on part of $\mathbb{R}$.*
   Suppose there exists a real number $B > T$ for which

10†:   *On $(T, B]$: $f \,\&\, g$ are diff'able, and g' is NevZ.*[♡1]

Recalling that $\mathbb{R}^\circledast = [-\infty, +\infty] \sqcup \{DNE\}$, define

$$\mathbf{L} := \lim_{x \searrow T} \frac{f(x)}{g(x)} \in \mathbb{R}^\circledast \qquad and$$

$$\mathbf{\Lambda} := \lim_{x \searrow T} \frac{f'(x)}{g'(x)} \in \mathbb{R}^\circledast \,.$$

[♡1]Courtesy (5), our g is zero at most once on $(T, B]$. Consequently, we can move B closer to T so that g is NevZ on $(T, B]$.

Suppose, as $x \searrow T$, that **either**

10_0:   *$g(x) \to 0$ and $f(x) \to 0$,*   **or**

10_∞:   *$g(x) \to \pm\infty$.*

If $\mathbf{\Lambda} \neq DNE$, then $\mathbf{L} = \mathbf{\Lambda}$.   ◊

Reduction to T finite. If $T = -\infty$, then define

$$\varphi(x) := f(\tfrac{-1}{x}) \quad \text{and}$$
$$\gamma(x) := g(\tfrac{-1}{x}) \,.$$

So $\lim_{x \searrow 0} \frac{\varphi}{\gamma}(x) = \mathbf{L}$. But the Chain Rule gives

$$\varphi'(x) = \tfrac{1}{x^2} \cdot f'(\tfrac{-1}{x}) \quad \text{and}$$
$$\gamma'(x) = \tfrac{1}{x^2} \cdot g'(\tfrac{-1}{x}) \,.$$

Thus $\lim_{x \searrow 0} \frac{\varphi'}{\gamma'}(x) = \mathbf{\Lambda}$. Proving L'H for $\boxed{\varphi, \gamma \text{ at } 0}$ will thus establish L'H for $\boxed{f, g \text{ at } -\infty}$.
   In the proofs below, we will take $(T, B] = (4, 6]$. □

Proof using (10_0). We may extend f by continuity so that f is cts on $[4, 6]$; hence $f(4) = 0$. Ditto for g.
   Each $x \in (4, 6]$, gives a point $\overset{\bullet}{x} \in (4, x)$ with

$$\frac{f'(\overset{\bullet}{x})}{g'(\overset{\bullet}{x})} \;\; \overset{\text{Cauchy-MVT}}{=\!=\!=\!=\!=} \;\; \frac{f(x) - f(4)}{g(x) - g(4)} \;\; \overset{\text{note}}{=\!=} \;\; \frac{f(x)}{g(x)} \,.$$

Sending $x \searrow 4$ forces $\overset{\bullet}{x} \to 4$.   ♦

Proof using (10_∞). Assume, say, $\mathbf{\Lambda}{=}7$. (The $\mathbf{\Lambda}{=}\pm\infty$ case just changes notation.) Fixing an $\varepsilon > 0$, ISTEstablish that

£1: $\displaystyle \limsup_{x \searrow 4} \frac{f}{g}(x) \quad \leq \quad 7 + \varepsilon \,.$

Inequality $\liminf \frac{f}{g} \geq 7 - \varepsilon$ will follow analogously.
   Take $B > 4$ so close to 4 that

£2: $\forall y \in (4, B] :$ $\left| \frac{f'}{g'}(y) - 7 \right| \leq \varepsilon \,.$

For each $x \in (4, B]$, the Cauchy-MVT gives a point $\overset{\bullet}{x} \in (x, B)$ with

£3: $\displaystyle \frac{f'(\overset{\bullet}{x})}{g'(\overset{\bullet}{x})} \;\; = \;\; \frac{f(B) - f(x)}{g(B) - g(x)} \;\; \overset{\text{note}}{=\!=} \;\; \frac{\frac{f(x)}{g(x)} - \frac{f(B)}{g(x)}}{1 - \frac{g(B)}{g(x)}} \,.$

And $\frac{f(B)}{g(x)}, \frac{g(B)}{g(x)} \to 0$, by (10_∞), as $x \searrow 4$.

Taking limsups in (£3) gives

∗: $$\limsup_{x \searrow 4} \tfrac{f'}{g'}(\overset{\bullet}{x}) \;\; = \;\; \limsup_{x \searrow 4} \tfrac{f}{g}(x) \,.$$

Now we ***don't know*** that $\overset{\bullet}{x} \to 4$. But we do know that each $\overset{\bullet}{x} \in (4, B]$. Thus LhS(∗) $\le 7 + \varepsilon$, by (£2). Hence (£1).   ♦

A remark on the above proof.

11: *Can* **L**=1 *yet* **Λ**=*DNE?*   Yes. For each sequence $x_n \searrow T$, nec. $\frac{f}{g}(x_n) \to 1$ and therefore $\frac{f'}{g'}(\overset{\bullet}{x_n}) \to 1$. Yet there *could* be a seq. $z_n \searrow T$ where, say,

$$\tfrac{f'}{g'}(z_n) \xrightarrow{n \to \infty} 927 \,. \quad \text{(I.e, some number} \neq 1.)$$

Yet it might that *every* $(\overset{\bullet}{x_n})_{n=1}^{\infty}$ sequence misses the dissenting $(z_n)_{n=1}^{\infty}$ sequence. Indeed, this is what happens in example (BttDB), above.   □

12: *Nifty application: Does $h' \to 0$?*   Given a differentiable function with *finite* limit, say $\lim_{x \to \infty} h(x) = 7$, suppose $\mathbf{L} := \lim_{x \to \infty} h'(x)$ **exists** in $\overline{\mathbb{R}}$. A picture suggests that $\mathbf{L}$ "must" be 0. Can we *prove* it by L'Hôpital's theorem?

Let $f := h \cdot \exp$ and $g := \exp$. So $\mathrm{Lim}(\frac{f}{g}) = 7$. But, $f' = [h' + h] \cdot \exp$. So $\mathrm{Lim}(\frac{f'}{g'}) = \mathbf{L} + 7$. Thus L'H applies to tell us that $7 = \mathbf{L} + 7$. Hence $\mathbf{L} = 0$. □

13: *FTCalculus.*   The conditions in (10_0) apply to

¥: $$\lim_{x \searrow 0} \frac{\int_0^x \sin(\sin(t)) \, \mathrm{d}t}{x^2} \,.$$

Since the **FTCalculus** applies to the numerator, **L'H** yields $\frac{f'}{g'}(x) = \frac{\sin(\sin(x))}{2x}$. Applying **L'H** again gives

$$\frac{f''}{g''}(x) \;\; = \;\; \frac{\cos(\sin(x)) \cdot \cos(x)}{2} \,.$$

As $\frac{f''}{g''}(0) = \frac{\cos(0) \cdot \cos(0)}{2} = \frac{1}{2}$, the limit in (13¥) is $\frac{1}{2}$.   □

Here is the famous example by ***O. Stolz***, published in 1871, generalized by ***R.P. Boas***, 1986.

Stolz example: (1871).   Let

$$f(x) := \int_0^x \cos(t)^2 \, \mathrm{d}t \overset{\text{note}}{=\!=\!=} \tfrac{1}{2} \cdot [x + \cos(x)\sin(x)] \,.$$

Easily $f(x) \nearrow \infty$ as $x \nearrow \infty$. Evidently
$$g(x) := f(x) \cdot \mathsf{e}^{\sin(x)}$$

goes $\to \infty$, since $\mathsf{e}^{\sin(x)}$ is bnded-below by $1/\mathsf{e}$, which is positive. Now $f' = \cos^2$. Thus

∗:
$$\begin{aligned} g'(x) \;\; &= \;\; \cos(x)^2 \cdot \mathsf{e}^{\sin(x)} + f(x) \cdot \mathsf{e}^{\sin(x)} \cdot \cos(x) \\ &= \;\; \cos(x) \cdot \mathsf{e}^{\sin(x)} \cdot [\cos(x) + f(x)] \,. \end{aligned}$$

Dividing produces the *to-zero-going* quantity

∗∗: $$\frac{f'}{g'}(x) \;\; = \;\; \frac{\cos(x)}{\mathsf{e}^{\sin(x)} \cdot [\cos(x) + f(x)]} \,, \qquad □$$

since **denom**$\to \infty$ and **numer** is bnded. Thus $\mathbf{\Lambda} = 0$. But $\mathbf{L}$ is the limit of $\frac{f(x)}{g(x)} \overset{\text{def}}{=\!=\!=} 1/\mathsf{e}^{\sin(x)}$, which is oscillatory, i.e **DNE**. *What went wrong?*

Give up? *Don't give up —examine the hypotheses of* **L'H** *with a finely-toothed comb.* ***If desperate, read further. . .***

In (∗), note that g' is zero **infinitely** often, due to the initial cos() factor. This contravenes the $(10\dagger)$ condition, despite the snow-job that I perpetrated in (∗∗).

In (∗), note that g' is zero **infinitely** often, due to the initial cos() factor. This contravenes the (10†) condition, despite the snow-job that I perpetrated in (∗∗).

In (∗), note that g' is zero **infinitely** often, due to the initial cos() factor. This contravenes the (10†) condition, despite the snow-job that I perpetrated in (∗∗).