2003 A6 Putnam Prob and Soln

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2003 Putnam

1: A6.2003. *For an $S \subset \mathbb{N}$, let $\mathbf{r}_S(n)$ denote the number of ordered pairs (s,t) with $\{s,t\} \subset S$, $s \neq t$, and $s+t = n$. [This (s,t) is an S-**representation** of n.] Is it possible to partition $\mathbb{N} = A \sqcup B$ in such a way that $\mathbf{r}_A(n) = \mathbf{r}_B(n)$ for all natnums n?* ♢

Soln A6.2003 (JK). Yes; the *Morse minimal sequence* is the unique solution.

Rename the sets from A,B to E,D (for Even and odD), and require that $E \ni 0$.

Uniqueness. Fix a soln $\mathbb{N} = E \sqcup D$. For each posint n, we will show that $\mathrm{E}' := E \cap [0\,..\,n)$ and $\mathrm{D}' := D \cap [0\,..\,n)$ determine where n goes. Recall $D \not\ni 0$, so $\mathbf{r}_D(n) = \mathbf{r}_{\mathrm{D}'}(n)$. Thus

$$\left[D \ni n\right] \iff \left[\mathbf{r}_{\mathrm{E}'}(n) \stackrel{*}{=} \mathbf{r}_{\mathrm{D}'}(n)\right].$$

Indeed, inequality in $(*)$, above, necessarily means that $\mathbf{r}_{\mathrm{D}'}(n)$ equals $2 + \mathbf{r}_{\mathrm{E}'}(n)$. So we give E two more representations, $(0,n)$ and $(n,0)$, by placing n in E.

Construction. For $n \in \mathbb{N}$, let $^{\#}n$ denote the number of 1-bits in BinaryNumeral(n). E.g $^{\#}(23) = 4$, since $23 = 10111$. Define

$$\text{1a:}\quad \begin{aligned} E &:= \left\{n \in \mathbb{N} \;\middle|\; {}^{\#}n \text{ is even}\right\} \\ D &:= \left\{n \in \mathbb{N} \;\middle|\; {}^{\#}n \text{ is odd}\right\}. \end{aligned}$$

Involution. Each subset $S \subset \mathbb{N}$ engenders

$$\widehat{S} \quad := \quad \{(s,t) \mid s,t \in S \text{ with } s \neq t\}.$$

Given natnums $s \neq t$, write them in binary as

$$\text{1b:}\quad \begin{aligned} s &= \mathsf{b}_K \mathsf{b}_{K-1} \ldots \mathsf{b}_4 \mathsf{b}_3 \mathsf{b}_2 \mathsf{b}_1 \mathsf{b}_0 \quad \text{and} \\ t &= \mathsf{c}_K \mathsf{c}_{K-1} \ldots \mathsf{c}_4 \mathsf{c}_3 \mathsf{c}_2 \mathsf{c}_1 \mathsf{c}_0\,. \end{aligned}$$

(Possibly b_K or c_K is 0.) Let ℓ be the smallest index st. $\mathsf{b}_\ell \neq \mathsf{c}_\ell$. Now *exchange* these two bits, defining a new pair (s',t'); for example, if $\ell = 2$ then

$$\text{1c:}\quad \begin{aligned} s' &= \mathsf{b}_K \mathsf{b}_{K-1} \ldots \mathsf{b}_4 \mathsf{b}_3 \mathbf{c}_2 \mathsf{b}_1 \mathsf{b}_0 \quad \text{and} \\ t' &= \mathsf{c}_K \mathsf{c}_{K-1} \ldots \mathsf{c}_4 \mathsf{c}_3 \mathbf{b}_2 \mathsf{c}_1 \mathsf{c}_0\,. \end{aligned}$$

The map $(s,t) \stackrel{f}{\longmapsto} (s',t')$ is an involution on $\widehat{\mathbb{N}}$. Moreover, it bijects $\widehat{E}$ with $\widehat{D}$. Lastly, f is sum-preserving; $s'+t' = s+t$. Thus $\mathbf{r}_E(n) = \mathbf{r}_D(n)$. ♦

Post provem. Define a 1-sided-infinite bit-sequence $\mathsf{b}_0\mathsf{b}_1\mathsf{b}_1\cdots$, where b_n is the parity of $^{\#}n$. This gives

$$0110\;1001\;1001\;0110\;1\cdots,$$

the ***Morse minimal sequence***. □

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