Liouville's Theorem: Calculus

Jonathan L.F. King
University of Florida, Gainesville FL 32611-2082, USA
squash@ufl.edu
Webpage http://squash.1gainesville.com/
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ABSTRACT: This is Liouville's proof of Liouville's thm on rational approximations of numbers.

Souvenir. The ***degree*** of an algebraic number $\boldsymbol{\alpha}$ is the degree of the smallest-degree non-zip intpoly having $\boldsymbol{\alpha}$ as a zero.

Whenever I write a rational, e.g $\frac{p}{q}$, the denominator q is *always positive.*

Warmup. Here is a classic theorem (due probably to Liouville).

1: Theorem. *Fix an $\boldsymbol{\alpha} \in \mathbb{R}$. Then*

†: $$\left|\boldsymbol{\alpha} - \tfrac{p}{q}\right| \quad\leq\quad 1/q^2$$

for a seq. of rationals $\frac{p}{q}$ with arbitrarily large q. ◊

Proof. WLOG $\boldsymbol{\alpha}$ is irrational. Take a large N and look at $0, \boldsymbol{\alpha}, 2\boldsymbol{\alpha}, \ldots, [N-1]\boldsymbol{\alpha}$, where we interprete these in the circle group, $\bmod 1$. By PHP (Pigeon-hole Principle), for some indices $j<k$ we must have the circle-gp distance $[\![k\boldsymbol{\alpha}, j\boldsymbol{\alpha}]\!] \leq 1/N$. With $q := k-j$, then, $[\![q\boldsymbol{\alpha}, 0]\!] = [\![k\boldsymbol{\alpha}, j\boldsymbol{\alpha}]\!] \leq 1/N$. I.e, for some integer p,

‡: $$|q\boldsymbol{\alpha} - p| \;\leq\; 1/N \overset{\text{note}}{\leq} 1/q\,.$$

[Indeed, this last inequality is strict, as $q \leq N-1$.] Dividing each side by q yields (†).

Write the dependancy as p_N and q_N. From (‡), $|\boldsymbol{\alpha} - \frac{p_N}{q_N}| \leq \frac{1}{N}$. So $\frac{p_N}{q_N} \to \boldsymbol{\alpha}$, an irrational, so the set $\{\frac{p_N}{q_N}\}_{N=1}^{\infty}$ is infinite. Thus q_N gets arbitrarily large. ♦

2: Liouville's Theorem. *Suppose $\boldsymbol{\alpha}$ is an irrational but algebraic number. Let $\mathfrak{D} := \operatorname{Deg}(\boldsymbol{\alpha}) \overset{\text{note}}{\geq} 2$. Then there exists a posreal C such that for all integers $q>0$ and p,*

3: $$\left|\boldsymbol{\alpha} - \frac{p}{q}\right| \quad\geq\quad C/q^{\mathfrak{D}}\,.$$ ◊

Proof. Let $f()$ be a deg-$\mathfrak{D}$ intpoly [necessarily $\mathbb{Q}$-irreducible] so that $f(\boldsymbol{\alpha}) = 0$. By continuity of f' [the derivative of f] there exists a small interval

$$J \quad:=\quad (\boldsymbol{\alpha} - \varepsilon,\ \boldsymbol{\alpha} + \varepsilon)$$

on which f' is bounded away from infinity. On J then, $1/f'$ is bounded away from zero; by $7/99$, say. Let $\boxed{C := \operatorname{Min}(\varepsilon, 7/99)}$; this is positive.

Consider an arbitrary rational, $\frac{p}{q}$. If $\frac{p}{q} \notin J$ then

$$\left|\boldsymbol{\alpha} - \frac{p}{q}\right| \;\geq\; \varepsilon \;\geq\; C \;\geq\; C/q^{\mathfrak{D}}\,.$$

So WLOG $\frac{p}{q} \in J$.

Since $f(\frac{p}{q})$ is not zero and f has integer coeffs,

4: $$\left|f\left(\tfrac{p}{q}\right)\right| \quad\geq\quad 1/q^{\mathfrak{D}}\,.$$

By the MVT (Mean Value Thm) there is a number ζ between $\boldsymbol{\alpha}$ and $\frac{p}{q}$ such that

$$[\boldsymbol{\alpha} - \tfrac{p}{q}] \cdot f'(\zeta) \quad=\quad f(\boldsymbol{\alpha}) - f(\tfrac{p}{q}) \overset{\text{note}}{=\!=} -f\left(\tfrac{p}{q}\right).$$

Taking absolute values, then dividing,

5: $$\left|\boldsymbol{\alpha} - \tfrac{p}{q}\right| \;=\; \tfrac{1}{|f'(\zeta)|} \cdot \left|f\left(\tfrac{p}{q}\right)\right| \;\geq\; \tfrac{1}{|f'(\zeta)|}\Big/ q^{\mathfrak{D}}\,.$$

But $\frac{p}{q} \in J$, so $\zeta \in J$. Hence (5) implies (3). ♦

Extension. Liouville's thm holds also when $\boldsymbol{\alpha}$ is rational and $\mathfrak{D} = 1$, as long as we only consider rationals $\frac{p}{q} \neq \boldsymbol{\alpha}$. Simply take J small enough that $\boldsymbol{\alpha}$ is the *only* zero of $f()$ on J; again (5) holds. □

Remark. Inequality (3) can be restated as

3′: $$\operatorname*{Infimum}_{\frac{p}{q} \in \mathbb{Q}} q^{\mathfrak{D}} \cdot \left|\boldsymbol{\alpha} - \frac{p}{q}\right| \quad>\quad 0\,.$$

This gives a criterion for transcendentality. A number with a sequence such that (6.1) is finite, is called a ***Liouville number***. □

6.0: Transcendental Theorem. *Consider a real* $\boldsymbol{\tau}$. *Suppose there exist* ∞*ly many rationals* p_N/q_N, *each **unequal** to* $\boldsymbol{\tau}$ *and with* $q_N \in [2\,..\,\infty)$, *st.*

6.1: $$\sup_{N\in\mathbb{Z}_+} [q_N]^N \cdot \left|\boldsymbol{\tau} - \frac{p_N}{q_N}\right|$$

is finite. Then $\boldsymbol{\tau}$ *is transcendental.* ♢

Proof. Suppose the supremum is 7. Fixing a posint $\mathfrak{D}$, could $\boldsymbol{\tau}$ have degree $\mathfrak{D}$? Well, for each $N \geq \mathfrak{D}$ note that

$$[q_N]^{\mathfrak{D}} \cdot \left|\boldsymbol{\tau} - \frac{p_N}{q_N}\right| \;\leq\; 7\Big/[q_N]^{N-\mathfrak{D}} \;\leq\; 7\Big/2^{N-\mathfrak{D}}\,.$$

The RhS goes to zero as $N\nearrow\infty$. This shows the failure of (3′); so $\boldsymbol{\tau}$ does *not* have degree-$\mathfrak{D}$. ♦

7: Example. *The following sum of factorial-powers is a Liouville number:*

$$\boldsymbol{\tau} := \sum_{j=1}^{\infty} 1/2^{[j!]}\,. \qquad ♢$$

Proof. Fix posint N, let $q := 2^{[N-1]!}$ and <u>define</u> p by $p/q := \sum_{j=1}^{N-1} 1/2^{j!}$; so $q = 2^{[N-1]!}$.

Now $2^{N!}\cdot|\boldsymbol{\tau} - \frac{p}{q}|$ equals

$$2^{N!}\cdot\sum_{j=N}^{\infty} 1/2^{j!} \;=\; \sum_{j=N}^{\infty} 1/2^{j!-N!}$$
$$\leq\; [1 + \tfrac{1}{2} + \tfrac{1}{4} + \ldots] = 2\,.$$

But $q^N = 2^{N!}$, so $q^N\cdot|\boldsymbol{\tau} - \frac{p}{q}| \leq 2$. Consequently, the number 2 dominates the (6.1) supremum. ♦

Transcendental sufficiency. A posint sequence $\vec{\boldsymbol{d}} = (d_1, d_2, d_3, \ldots)$ is ***good*** if $d_n\nearrow\infty$ and

$*$: $$d_1 \bullet\!\!| \; d_2 \bullet\!\!| \; d_3 \bullet\!\!| \;\cdots$$

Define the N^{th}***-tail*** by $\text{Tail}_N(\vec{\boldsymbol{d}}) := \sum_{j=N+1}^{\infty}\frac{1}{d_j}$.

8: Liouville sufficiency lemma. *Consider a good* $\vec{\boldsymbol{d}}$ *such that*

$$[d_N]^N \cdot \text{Tail}_N(\vec{\boldsymbol{d}}) \;\xrightarrow{N\to\infty}\; 0\,,$$

or, weaker, simply has finite limit-supremum. Then $\boldsymbol{\tau} := \sum_{j=1}^{\infty}\frac{1}{d_j}$ *is transcendental.* ♢

Pf. Value $p_N := d_N\cdot\sum_{j=1}^{N}\frac{1}{d_j}$ is an integer, courtesy ($*$). Thus $\boldsymbol{\tau} - \frac{p_N}{d_N} = \text{Tail}_N(\vec{\boldsymbol{d}})$. Hence

$$[d_N]^N\cdot\left|\boldsymbol{\tau} - \frac{p_N}{d_N}\right| \;=\; [d_N]^N\cdot\text{Tail}_N(\vec{\boldsymbol{d}})\,.$$

Now apply (6.1). ♦

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