Linear Recurrence using matrices

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See also *Problems/NumberTheory/fibonacci.latex*

Notation. Use "$\overset{r}{\sim}$" for row-equivalence: I.e, $K{\times}N$ matrices $\mathsf{M} \overset{r}{\sim} \mathsf{M}'$ IFF we can get from M to M' using row operations. Use "$\overset{c}{\sim}$" for column-equivalence.

Two $N{\times}N$ matrices B & C are ***similar***, or ***conjugate*** to each other, if *there exists* an invertible matrix U such that

$$\mathsf{U}^{-1} \cdot \mathsf{B} \cdot \mathsf{U} \quad = \quad \mathsf{C}\,.$$

Use $\mathsf{B} \overset{\text{sim}}{\sim} \mathsf{C}$ for the similarity equiv-relation.

Call a matrix OTForm $s\mathbf{I} \overset{\text{note}}{=\!=} \left[\begin{smallmatrix} s & 0 \\ 0 & s \end{smallmatrix}\right]$, a ***dilation***; its action on the plane is simply to scale-uniformly by factor s. *Every* matrix commutes with $\mathbf{I}$, and so:

1: *A dilation is only conjugate to itself.*

I.e, for invertible U, necessarily $\mathsf{U}^{-1} \cdot s\mathbf{I} \cdot \mathsf{U} = s\mathbf{I}$.

For T a square matrix (or a trn from a vectorspace to itself) and a complex number α, define the subspace

$$\mathbb{E}_{\mathsf{T},\alpha} \;:=\; \{\text{vectors } \mathbf{v} \mid \mathsf{T}\mathbf{v} = \alpha\mathbf{v}\}\,.$$

Saying that "*α is a T-eigenvalue*" is the same as saying that $\mathrm{Dim}(\mathbb{E}_{\mathsf{T},\alpha}) \geq 1$.

Linear recurrence

A fibonacci-like seq $\vec{\mathbf{z}} := (z_n)_{n=-\infty}^{\infty}$ is specified using (complex) numbers $\mathcal{S}$ and $\mathcal{P}$, with $\mathcal{P}$ non-zero[♡1], by

2: $$z_{n+2} \;:=\; \mathcal{S}z_{n+1} - \mathcal{P}z_n\,,$$

and some initial condition (z_1, z_0). With

$$\mathsf{G} \;:=\; \left[\begin{smallmatrix} \mathcal{S} & -\mathcal{P} \\ 1 & 0 \end{smallmatrix}\right], \quad \text{then } \left[\begin{smallmatrix} z_{n+1} \\ z_n \end{smallmatrix}\right] = \mathsf{G}^n \cdot \left[\begin{smallmatrix} z_1 \\ z_0 \end{smallmatrix}\right],$$

for each integer n. We want to diagonalize G.

[♡1] Actually, recurrence (2) can be run backwards, giving z_n values when n is negative, as soon as $(\mathcal{S},\mathcal{P}) \neq (0,0)$. However, the $\mathcal{P}{\neq}0$ case is the interesting case.

Its char-poly is $f(x) = f_{\mathsf{G}}(x) := x^2 - \mathcal{S}x + \mathcal{P}$. Factor this f as $f(x) = [x - \boldsymbol{\alpha}][x - \boldsymbol{\beta}]$, with $\boldsymbol{\alpha},\boldsymbol{\beta} \in \mathbb{C}$. Equating coeffs in the polynomial gives

2a: $$\boldsymbol{\alpha} + \boldsymbol{\beta} \;=\; \mathcal{S}\,; \qquad \text{(Sum)}$$
$$\boldsymbol{\alpha} \cdot \boldsymbol{\beta} \;=\; \mathcal{P}\,. \qquad \text{(Product)}$$

Since $\mathcal{P} \neq 0$, necessarily $\boxed{\boldsymbol{\alpha} \neq 0}$ and $\boxed{\boldsymbol{\beta} \neq 0}$. Each of $\boldsymbol{\alpha}, \boldsymbol{\beta}$ is a root of f, hence

$$\boldsymbol{\alpha}^2 \;=\; \mathcal{S}\boldsymbol{\alpha} - \mathcal{P} \quad \text{and} \quad \boldsymbol{\beta}^2 \;=\; \mathcal{S}\boldsymbol{\beta} - \mathcal{P}\,.$$

$$\textit{Specifically:} \quad \boldsymbol{\alpha},\boldsymbol{\beta} \;=\; \tfrac{1}{2}\big[\mathcal{S} \pm \sqrt{\mathcal{S}^2 - 4\mathcal{P}}\,\big].$$

Finding an $\boldsymbol{\alpha}$-eigenvector. An $\boldsymbol{\alpha}$-eVec (for G) is an element of $\mathrm{Nul}(\mathsf{G} - \boldsymbol{\alpha}\mathbf{I})$. Applying row operations,

2b: $$\mathsf{G} - \boldsymbol{\alpha}\mathbf{I} \;=\; \begin{bmatrix} \mathcal{S} - \boldsymbol{\alpha} & -\mathcal{P} \\ 1 & -\boldsymbol{\alpha} \end{bmatrix} \overset{r}{\sim} \begin{bmatrix} 1 & -\boldsymbol{\alpha} \\ \mathcal{S} - \boldsymbol{\alpha} & -\mathcal{P} \end{bmatrix} \overset{r}{\sim} \begin{bmatrix} 1 & -\boldsymbol{\alpha} \\ 0 & 0 \end{bmatrix}.$$

This last $\overset{r}{\sim}$ requires **<u>no</u>** computation, since the rows *must* be linearly-dependent (since $\boldsymbol{\alpha}$ is an eigenvalue of G).

This last matrix has one free column, and evidently multiplies $\left[\begin{smallmatrix} \boldsymbol{\alpha} \\ 1 \end{smallmatrix}\right]$ to the zero-vector. So

2c: *The singleton $\left\{\left[\begin{smallmatrix} \boldsymbol{\alpha} \\ 1 \end{smallmatrix}\right]\right\}$ is a basis for the $\boldsymbol{\alpha}$-eigenspace of G, which is one-dimensional.*

Instead of row-ops to show that $\mathbb{E}_{\mathsf{G},\boldsymbol{\alpha}}$ (the $\boldsymbol{\alpha}$-eigenspace of G) is one-dim'al, we could have argued as follows: *Were $\mathbb{E}_{\mathsf{G},\boldsymbol{\alpha}}$ two-dim'al, then $\mathsf{G} \overset{\text{sim}}{\sim} \left[\begin{smallmatrix} \boldsymbol{\alpha} & 0 \\ 0 & \boldsymbol{\alpha} \end{smallmatrix}\right]$. By (1), then, G would <u>equal</u> $\left[\begin{smallmatrix} \boldsymbol{\alpha} & 0 \\ 0 & \boldsymbol{\alpha} \end{smallmatrix}\right]$ —which it doesn't!*

When is $\boldsymbol{\alpha} = \boldsymbol{\beta}$? This happens when the discriminant of f is zero. Its discrim is $[\text{-}\mathcal{S}]^2 - 4 \cdot 1 \cdot \mathcal{P}$, i.e $[\boldsymbol{\alpha}^2 + \boldsymbol{\beta}^2 + 2\boldsymbol{\alpha}\boldsymbol{\beta}] - 4\boldsymbol{\alpha}\boldsymbol{\beta}$. Hence

3a: $$\mathrm{Discr}(f) \;=\; \mathcal{S}^2 - 4\mathcal{P} \;\overset{\text{note}}{=\!=}\; [\boldsymbol{\alpha} - \boldsymbol{\beta}]^2\,.$$

Since (2c) also applies to eigenvalue $\boldsymbol{\beta}$, we conclude:

3b: *Matrix $\mathsf{G}{=}\left[\begin{smallmatrix} \mathcal{S} & -\mathcal{P} \\ 1 & 0 \end{smallmatrix}\right]$ is diagonalizable IFF G has distinct eigenvalues; i.e $\mathcal{S}^2 \neq 4\mathcal{P}$.*

Distinct eigenvalues

When $\boldsymbol{\alpha} \neq \boldsymbol{\beta}$, our (2c) implies that G is conjugate to diagonal-matrix

$$4: \quad \mathsf{D} := \begin{bmatrix} \boldsymbol{\alpha} & 0 \\ 0 & \boldsymbol{\beta} \end{bmatrix} \quad \text{via matrix}$$
$$\mathsf{U} := \begin{bmatrix} \boldsymbol{\alpha} & \boldsymbol{\beta} \\ 1 & 1 \end{bmatrix}. \quad \text{Note} \quad \mathsf{U}^{-1} = \frac{1}{\boldsymbol{\alpha}-\boldsymbol{\beta}} \cdot \begin{bmatrix} 1 & -\boldsymbol{\beta} \\ -1 & \boldsymbol{\alpha} \end{bmatrix}.$$

We populated U with evecs, using the same order of evals in D. One can check that $\boxed{\mathsf{U}^{-1}\mathsf{G}\mathsf{U} = \mathsf{D}}$. Hence $\mathsf{G} = \mathsf{U}\mathsf{D}\mathsf{U}^{-1}$. So for each integer n,

$$5a: \quad \mathsf{G}^n = \frac{1}{\boldsymbol{\alpha}-\boldsymbol{\beta}} \begin{bmatrix} \boldsymbol{\alpha} & \boldsymbol{\beta} \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{\alpha}^n & 0 \\ 0 & \boldsymbol{\beta}^n \end{bmatrix} \cdot \begin{bmatrix} 1 & -\boldsymbol{\beta} \\ -1 & \boldsymbol{\alpha} \end{bmatrix}.$$

Multiplying from the right by initial-condition $\begin{bmatrix} z_1 \\ z_0 \end{bmatrix}$ will produce $\begin{bmatrix} z_{n+1} \\ z_n \end{bmatrix}$. Extracting the z_n, we get the nice formula

$$5b: \quad z_n = \frac{1}{\boldsymbol{\alpha}-\boldsymbol{\beta}} \cdot \Big[[z_1 - \boldsymbol{\beta} z_0]\boldsymbol{\alpha}^n - [z_1 - \boldsymbol{\alpha} z_0]\boldsymbol{\beta}^n\Big].$$

This formula is symmetric in $\boldsymbol{\alpha}$ & $\boldsymbol{\beta}$, as it must be.

Equal eigenvalues

Let's take a look at the $\boldsymbol{\alpha}{=}\boldsymbol{\beta}$ case. This means that

$$6: \quad \mathsf{G} = \begin{bmatrix} 2\boldsymbol{\beta} & -\boldsymbol{\beta}^2 \\ 1 & 0 \end{bmatrix}.$$

While we can't conjugate G to a *diagonal* matrix, we <u>can</u> conjugate to its ***Jordan canonical form***. The JCF of (6) is

$$\mathsf{J} = \mathsf{J}_{\mathsf{G}} := \begin{bmatrix} \boldsymbol{\beta} & 1 \\ 0 & \boldsymbol{\beta} \end{bmatrix}.$$

Mysteriously pulling the below C from a hat, multiplication verifies that

$$7: \quad \mathsf{J} = \mathsf{C}^{-1}\mathsf{G}\mathsf{C}, \quad \text{where} \quad \mathsf{C} := \begin{bmatrix} \boldsymbol{\beta} & 1 \\ 1 & 0 \end{bmatrix}$$

and $\mathsf{C}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -\boldsymbol{\beta} \end{bmatrix}$. This will yield a formula for z_n.

Induction on n shows that

$$\mathsf{J}^n := \boldsymbol{\beta}^{n-1} \cdot \begin{bmatrix} \boldsymbol{\beta} & n \\ 0 & \boldsymbol{\beta} \end{bmatrix}, \quad \text{for each } n \in \mathbb{Z}.$$

As before, $\mathsf{G} = \mathsf{C}\mathsf{J}\mathsf{C}^{-1}$ so $\mathsf{G}^n = \mathsf{C}\mathsf{J}^n\mathsf{C}^{-1}$. That is,

$$8a: \quad \mathsf{G}^n = \boldsymbol{\beta}^{n-1} \cdot \begin{bmatrix} \boldsymbol{\beta} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\beta} & n \\ 0 & \boldsymbol{\beta} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -\boldsymbol{\beta} \end{bmatrix} = \boldsymbol{\beta}^{n-1} \cdot \begin{bmatrix} [1+n]\boldsymbol{\beta} & -n\boldsymbol{\beta}^2 \\ n & [1-n]\boldsymbol{\beta} \end{bmatrix}.$$

Multiplying by $\begin{bmatrix} z_1 \\ z_0 \end{bmatrix}$; the bottom entry in the resulting column-vector is

$$8b: \quad z_n = \Big[nz_1 + [1-n]\boldsymbol{\beta} z_0\Big] \cdot \boldsymbol{\beta}^{n-1} = \Big[[z_1 - \boldsymbol{\beta} z_0]n + \boldsymbol{\beta} z_0\Big] \cdot \boldsymbol{\beta}^{n-1}.$$

Addendum. We could have derived (8b) from (5b), by sending $\boldsymbol{\alpha} \to \boldsymbol{\beta}$, then using l'Hôpital's rule:

Derivative $\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\alpha}}[\boldsymbol{\alpha} - \boldsymbol{\beta}]$ is 1. And

$$\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\alpha}}\Big[[z_1 - \boldsymbol{\beta} z_0]\boldsymbol{\alpha}^n - [z_1 - \boldsymbol{\alpha} z_0]\boldsymbol{\beta}^n\Big]$$
$$= [z_1 - \boldsymbol{\beta} z_0]{\cdot}n\boldsymbol{\alpha}^{n-1} - {-z_0}\boldsymbol{\beta}^n$$
$$= [z_1 - \boldsymbol{\beta} z_0]{\cdot}n\boldsymbol{\alpha}^{n-1} + z_0\boldsymbol{\beta}^n.$$

Applying $\lim_{\boldsymbol{\alpha}\to\boldsymbol{\beta}}$ to this last expression, yields

$$[z_1 - \boldsymbol{\beta} z_0]{\cdot}n\boldsymbol{\beta}^{n-1} + z_0\boldsymbol{\beta}^n \overset{\text{note}}{=\!=} \text{RhS}(8b). \quad \square$$

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