Projections on a line : LinearAlg

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Problem

Let $\mathsf{P} = \mathsf{P}_{30°}$ be the 2×2 matrix whose lefthand action is ortho-projection on the angle=30° line through the origin. Compute P.

Support your answer in three conceptually-different ways.

Strategy

Step S0. A simplifying notation: Use $c := \cos(30°)$ and $s := \sin(30°)$.

Step S1. Use similar triangles to compute $\mathsf{P}\cdot\mathbf{e}_1$; this product gives the first column of P. Use similar triangles to compute $\mathsf{P}\cdot\mathbf{e}_2$; this equals the second column of P.

Step S2. Partial verification of our result: Each vector **v** on the 30° line should be *fixed* (not moved) by the projection. So $\mathsf{P}\cdot\mathbf{v}$ better equal **v**.

Step S3. "Variation of parameters": Generalize the result of Step 1, replacing "30°" by a general angle ω. Easily

$$*:\quad \mathsf{P}_{0°} = \begin{bmatrix}1&0\\0&0\end{bmatrix} \quad \textit{and} \quad \mathsf{P}_{90°} = \begin{bmatrix}0&0\\0&1\end{bmatrix}.$$

Now we will check that our P_ω formula gives these two matrices when ω is set equal to 0° and 90°.

Step S4. "The Dictionary Idea": Instead, let us obtain $\mathsf{P}_{30°}$ by *conjugation* by rotation matrix $\mathsf{R}_{30°}$. So we check that

$$\mathsf{P}_\omega \stackrel{?}{=} \mathsf{R}_\omega \cdot \mathsf{P}_{0°} \cdot \mathsf{R}_{-\omega}\ .$$

(Aside: Here, given two $N{\times}N$ matrices B and invertible D, let "B conjugated by D" mean $\mathsf{D}\cdot\mathsf{B}\cdot\mathsf{D}^{-1}$.)

Applying the Strategy

Doing S1. Similar triangles yields

$$1:\quad \mathsf{P} = \begin{bmatrix}c\cdot c & c\cdot s\\ c\cdot s & s\cdot s\end{bmatrix}.$$ ♦

Doing S2. The unit-vectors on the 30° line are ±**v**, where $\mathbf{v} := \left[\begin{smallmatrix}c\\s\end{smallmatrix}\right]$. Define $\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right] := \mathsf{P}\cdot\left[\begin{smallmatrix}c\\s\end{smallmatrix}\right]$. The beloved Pythagorean theorem yields that

$$2:\quad x = [c{\cdot}c]{\cdot}c + [c{\cdot}s]{\cdot}s = c\cdot[c^2+s^2] \stackrel{\text{Pythag}}{=\!=} c.$$

Similarly $y = s$. Thus we see that $\mathsf{P}\cdot\mathbf{v}$ indeed equals **v**. ♦

Doing S3. The derivation of (1) applies when 30° is replaced by a general angle ω. Redefining $c := \cos$ and $s := \sin$ thus gives

$$3:\quad \mathsf{P}_\omega = \begin{bmatrix}c(\omega)\cdot c(\omega) & c(\omega)\cdot s(\omega)\\ c(\omega)\cdot s(\omega) & s(\omega)\cdot s(\omega)\end{bmatrix}.$$

So $\mathsf{P}_{0°}$ is $\left[\begin{smallmatrix}1\cdot1 & 1\cdot0\\ 1\cdot0 & 0\cdot0\end{smallmatrix}\right]$, which indeed equals $\left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right]$, the $\mathsf{P}_{0°}$ from (∗). The check for $\mathsf{P}_{90°}$ similarly works. Note that we could also have used $\mathsf{P}_{45°}$ as an easy check. ♦

Doing S4. Letting c and s denote $\cos(\omega)$ and $\sin(\omega)$, recall our rotation matrix $\mathsf{R}_\omega = \left[\begin{smallmatrix}c & -s\\ s & c\end{smallmatrix}\right]$. Since cos() is an even fnc, and sin() is odd, automatically $\mathsf{R}_{-\omega} = \left[\begin{smallmatrix}c & s\\ -s & c\end{smallmatrix}\right]$. Multiplying the matrices $\mathsf{R}_\omega\cdot\mathsf{P}_{0°}\cdot\mathsf{R}_{-\omega}$ indeed yields RhS(1), as hoped. ♦

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