The Laplace Transform: Calculus

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ABSTRACT: This first gives a defn of *exponential order* which is better adapted to convolution. Following, is a discussion of the "tapping on a bell" problem; one text called this "soldiers marching on a bridge". Both interpretations need a grain-of-salt...

Prelims. On a (possibly infinite) interval $J \subset \mathbb{R}$, a function $f: J \to \mathbb{C}$ is ***locally-integrable*** if, for each bounded subinterval $[a, b] \subset J$, integral $\int_a^b f$ exists and is finite. A *sufficient* (but not necessary) condition is that on each $[a, b]$, our f is bounded with only finitely-many discontinuities.

For a fnc $f: [0, \infty) \to \mathbb{C}$ and complex number s we define the "Laplace transform of f", evaluated at s,

1: $$\widehat{f}(s) = [\mathcal{L}(f)](s) := \int_0^\infty \mathrm{e}^{-st} \cdot f(t) \cdot \mathrm{d}t\,,$$

for those values of s where this integral exists.

For complex numbers α and β, let $\alpha \succ \beta$ mean $\mathrm{Re}(\alpha) > \mathrm{Re}(\beta)$. For a real $\boldsymbol{\mu}$, say that "f has ***exponential order*** $\boldsymbol{\mu}$", written $\boxed{f \in \mathrm{Ord}(\boldsymbol{\mu})}$, if $f: [0, \infty) \to \mathbb{C}$ and f is locally-integrable *and*

†: $$\forall Q > \boldsymbol{\mu}: \quad \lim_{t \to \infty} |f(t)| / \mathrm{e}^{Q \cdot t} = 0\,.$$

One can replace (†) by the seemingly weaker

‡: $$\forall Q > \boldsymbol{\mu}: \quad \limsup_{t \to \infty} |f(t)| / \mathrm{e}^{Q \cdot t} < \infty\,,$$

but they are [exercise] equivalent.

2: Lemma. *Consider an $f \in \mathrm{Ord}(\boldsymbol{\mu})$. Then $\widehat{f}(s)$ exists for each s with $\mathrm{Re}(s) > \boldsymbol{\mu}$. Indeed the integrand in (1) is absolutely integrable.* ♢

Proof. Fix s with $x := \mathrm{Re}(s) > \boldsymbol{\mu}$, then pick Q with $x > Q > \boldsymbol{\mu}$. Our t is real, so $|\mathrm{e}^{-st}| = \mathrm{e}^{-xt}$. Hence the integrand in (1) is eventually bounded,

$$\left|\mathrm{e}^{-st} \cdot f(t)\right| < \mathrm{e}^{-xt} \cdot \mathrm{e}^{Qt} \overset{\text{note}}{=\!=} \mathrm{e}^{-[x-Q]t}\,,$$

once t is large enough. Since $x - Q$ is positive, this last is integrable over $t \in [0, \infty)$. ♦

We need the following standard tool, to prove a uniqueness result.

3a: Identically-zero lemma. *On a closed bounded interval $J := [b, c] \subset \mathbb{R}$, consider continuous functions $h, G: J \to \mathbb{C}$.*

i: *Suppose $h \geqslant 0$. If $\int_J h = 0$, then $h \equiv 0$, i.e, h is **identically-zero**.* [Exer:]

ii: *If G is real-valued and $\int_J [G \cdot G] = 0$, then $G \equiv 0$.* [Exer: Set $h := G \cdot G$, etc.]

iii: *Suppose*

3b: $$\forall n \in \mathbb{N}: \quad \int_J x^n \cdot G(x)\,\mathrm{d}x = 0\,.$$

Then G is identically-zero on J. ♢

Proof of (iii). Splitting G into real and imaginary parts, WLOG G is real-valued. Stmt (3b) implies, for each polynomial p, that $\int_J [p \cdot G]$ is zero.

By the Weierstrass Approximation Thm, there is a sequence of (real) polynomials, so that $p_k \overset{k \to \infty}{\longrightarrow} G$ *uniformly*. Consequently,

$$\int_J [G \cdot G] \overset{\text{Exer.}}{=\!=} \lim_{k \to \infty} \int_J [p_k \cdot G] = \lim_{k \to \infty} 0 = 0\,. \ ♦$$

3c: Uniqueness Thm. *Consider fncs $g, \gamma \in \mathrm{Ord}(\boldsymbol{\mu})$. Suppose their Laplace transforms, $\widehat{g}$ and $\widehat{\gamma}$, agree on some real interval $\mathbf{I} := [s_0, \infty)$. If g and γ are continuous, then $g = \gamma$.* ♢

[Proof is in notes, commented out.]

Convolution. Recall that the (one-sided) ***convolution*** of two (locally-integrable) functions $f, g\colon [0,\infty)\to\mathbb{C}$, is the function

4a: $$[f \circledast g](t) \;:=\; \int_0^t f(t-v)\cdot g(v)\,\mathrm{d}v\,.$$

4b: Lemma. *Suppose $f, g \in \mathrm{Ord}(\boldsymbol{\mu})$. Then the convolution $f \circledast g \in \mathrm{Ord}(\boldsymbol{\mu})$.* ♢

[Proof is in notes, commented out.]

4c: Lap-of-Convolve. $\Big[\mathcal{L}(f \circledast g)\Big](s) \;=\; \widehat{f}(s)\cdot\widehat{g}(s)$. ♢

Proof. Define $\mathbf{1}(\mathsf{true}) := 1$ and $\mathbf{1}(\mathsf{false}) := 0$. We can now write RhS$(4a)$ as

$$\int_0^\infty \mathbf{1}(v \leqslant t)\cdot f(t-v)\cdot g(v)\,\mathrm{d}v\,.$$

Hence $\Big[\mathcal{L}(f \circledast g)\Big](s)$ equals

$$\int_0^\infty \mathsf{e}^{-st}\int_0^\infty \mathbf{1}(v \leqslant t)\cdot f(t-v)\cdot g(v)\,\mathrm{d}v\,\mathrm{d}t\,.$$

Under mild conditions[♡1] on f and g, we can reverse the integrals, giving that $\widehat{[f \circledast g]}(s)$ equals

$$\int_0^\infty g(v)\int_0^\infty \mathbf{1}(v \leqslant t)\cdot \mathsf{e}^{-st}\cdot f(t-v)\,\mathrm{d}t\,\mathrm{d}v\,.$$

The inner integral can be written as

¥: $$\int_v^\infty \mathsf{e}^{-st}\cdot f(t-v)\,\mathrm{d}t\,.$$

CoV $x = t-v$ traverses $v-v \nearrow t-v \nearrow \infty - v = \infty$, and $\mathrm{d}x = \mathrm{d}t$. So (¥) equals

$$\begin{aligned}\int_0^\infty \mathsf{e}^{-s[x+v]}\cdot f(x)\,\mathrm{d}x \;&=\; \mathsf{e}^{-sv}\int_0^\infty \mathsf{e}^{-sx}\cdot f(x)\,\mathrm{d}x \\ &=\; \mathsf{e}^{-sv}\cdot \widehat{f}\,(s)\,.\end{aligned}$$

Thus $\Big[\mathcal{L}(f \circledast g)\Big](s)$ equals

$$\int_0^\infty g(v)\cdot \mathsf{e}^{-sv}\cdot\widehat{f}\,(s)\,\mathrm{d}v \;\stackrel{\text{note}}{=\!=}\; \widehat{f}\,(s)\cdot\widehat{g}\,(s)\,.$$ ♦

[♡1] See Fubini–Tonelli theorem.

5a: Deriv-of-Lap. *Fix a natnum N. Taking the N^{th} derivative,*

$$\left[\frac{\mathrm{d}}{\mathrm{d}s}\right]^N \widehat{f}(s) \;=\; \mathcal{L}\Big([-t]^N \cdot f(t)\Big)(s) \;\underset{\text{note}}{=\!=}\; [-1]^N \cdot \mathcal{L}\Big(t^N \cdot f(t)\Big)(s)\,. \qquad \diamond$$

5b: Integral-of-Lap. *For real s with $s > \boldsymbol{\mu}$,*

$$\int_s^\infty \widehat{f}(u)\,\mathrm{d}u \;=\; \mathcal{L}\Big(\frac{f(t)}{t}\Big)(s)\,. \qquad \diamond$$

5c: Lap-of-Deriv. *Suppose f differentiable. Then*

$$\widehat{f'}(s) \;=\; \Big[s \cdot \widehat{f}(s)\Big] - f(0)\,.$$

For $N \in \mathbb{N}$, suppose that an N-times differentiable f, has $f, f', \dots, f^{(N-1)} \in \mathrm{Ord}(\boldsymbol{\mu})$. Then

$$\widehat{f^{(N)}}(s) \;=\; \Big[s^N \cdot \widehat{f}(s)\Big] - \sum_{\substack{j+k=N-1 \\ j,k \in \mathbb{N}}} s^j \cdot \Big[f^{(k)}(0)\Big]\,,$$

where the sum is over all ordered-pairs (j, k) of natnums. $\diamond$

Pf of (5a), *Deriv-of-Lap.* With appropriate conditions on f, we can differentiate under the integral sign in (1), applying $\frac{\mathrm{d}}{\mathrm{d}s}$, to get the $N{=}1$ case of (5a). Now induct on N. ♦

Pf of (5b), *Integral-of-Lap.* For t positive, note

$$*: \quad \int_s^\infty e^{-ut}\,\mathrm{d}u \;=\; \frac{1}{t} \cdot e^{-ut}\Big|_{u=\infty}^{u=s} \;=\; \frac{1}{t} \cdot [e^{-st} - e^{-\infty}] \;=\; \frac{1}{t} \cdot e^{-st}\,.$$

Applying the definition, $\int_s^\infty \widehat{f}(u)\,\mathrm{d}u$ equals

$$\int_s^\infty \Big[\int_0^\infty e^{-ut} \cdot f(t) \cdot \mathrm{d}t\Big]\,\mathrm{d}u \;=\; \int_0^\infty f(t) \int_s^\infty e^{-ut}\,\mathrm{d}u\,\mathrm{d}t \;\underset{\text{by }(*)}{=\!=}\; \int_0^\infty f(t) \cdot \frac{1}{t} \cdot e^{-st}\,\mathrm{d}t\,.$$

This latter is the defn of $\mathcal{L}\Big(\frac{f(t)}{t}\Big)(s)$. ♦

Pf of (5c), *Lap-of-Deriv.* Our $\widehat{f'}(s)$ is the limit, as $M \nearrow \infty$, of

$$\int_0^M \underbrace{e^{-st}}_{u} \cdot \underbrace{f'(t)\,\mathrm{d}t}_{\mathrm{d}v}\,.$$

Integrating by parts, this equals

$$*: \quad \underbrace{e^{-st}}_{u} \cdot \underbrace{f(t)}_{v}\Big|_{t=0}^{t=M} - \int_0^M \underbrace{f(t)}_{v} \cdot \underbrace{[-s]e^{-st}\,\mathrm{d}t}_{\mathrm{d}u}$$
$$= \Big[s \cdot \int_0^M f(t) \cdot e^{-st}\,\mathrm{d}t\Big] - \frac{f(t)}{e^{st}}\Big|_{t=M}^{t=0}\,.$$

This last term equals $\frac{f(0)}{1} - \frac{f(M)}{e^{sM}} = f(0) - \frac{f(M)}{e^{sM}}$. If $\mathrm{Re}(s) > \boldsymbol{\mu}$, then $\frac{f(M)}{e^{sM}} \to 0$. as $M \nearrow \infty$. Sending $M \nearrow \infty$ thus sends $(*)$ to $\Big[s \cdot \widehat{f}(s)\Big] - f(0)$, as desired.

Finally, the formula for $\widehat{f^{(N)}}$ follows by induction on N. ♦

6: Lap-of-Exp-prod. *Fix $B \in \mathbb{C}$. Then*

$$\mathcal{L}\Big(e^{Bt} \cdot f(t)\Big)(s) \;=\; \widehat{f}(s - B)\,,$$

whenever $s \succ B + \boldsymbol{\mu}$. $\diamond$

Proof. By defn, $\mathcal{L}\Big(e^{Bt} \cdot f(t)\Big)(s)$ equals

$$\int_0^\infty e^{-st} e^{Bt} f(t)\,\mathrm{d}t \;=\; \int_0^\infty e^{-[s-B]t} f(t)\,\mathrm{d}t\,.$$

This last integral converges once $\mathrm{Re}(s - B) > \boldsymbol{\mu}$, i.e, once $s \succ B + \boldsymbol{\mu}$. ♦

Examples

Below, $B \in \mathbb{C}$ and $F \in \mathbb{R}$. Easily $[\mathcal{L}(1)](s)$ equals $1/s$. From Deriv-of-Lap, then, $[-1]^N \cdot \mathcal{L}\big(t^N\big)(s)$ equals $[\frac{\mathrm{d}}{\mathrm{d}s}]^N\big(\widehat{1}(s)\big) = [\frac{\mathrm{d}}{\mathrm{d}s}]^N\big(\frac{1}{s}\big)$. Thus

7a: $$\mathcal{L}\big(t^N\big)(s) \;=\; \frac{N!}{s^{N+1}}\,.$$

From Lap-of-Exp-prod, note,

$*$: $$[\mathcal{L}(\mathrm{e}^{Bt})](s) \;=\; \frac{1}{s-B}\,.$$

Recall $\cos(Ft) = \frac{1}{2}[\mathrm{e}^{\mathbf{i}Ft} + \mathrm{e}^{-\mathbf{i}Ft}]$. So $(*)$ gives

$$2 \cdot \mathcal{L}\big(\cos(Ft)\big)(s) \;=\; \tfrac{1}{s-\mathbf{i}F} + \tfrac{1}{s+\mathbf{i}F} \;=\; \frac{2s}{s^2+F^2}\,.$$

Dividing both sides by 2,

7b: $$\mathcal{L}\big(\cos(Ft)\big)(s) \;=\; \frac{s}{s^2+F^2}\,, \quad \textit{and similarly}$$
$$\mathcal{L}\big(\sin Ft\big)(s) \;=\; \frac{F}{s^2+F^2}\,.$$

On $[0,\infty)$, $\sin()$ is bounded, so has a Lap-xform there. By (7b), $\widehat{\sin}(z) = \frac{1}{z^2+1} \overset{\text{recall}}{=\!=} \operatorname{atan}'(z)$. So

$$\begin{aligned}\int_0^\infty \widehat{\sin}\,(z)\,\mathrm{d}z \;&=\; \lim_{U\nearrow\infty}\int_0^U \widehat{\sin}\,(z)\,\mathrm{d}z \\ &=\; \Big[\lim_{U\nearrow\infty}\operatorname{atan}(U)\Big] - \operatorname{atan}(0) \\ &=\; \frac{\pi}{2} - 0 \;=\; \frac{\pi}{2}\,.\end{aligned}$$

Skipping the effort it takes to *justify* applying (5b) at $s{=}0$, we get that

$$\int_0^\infty \underbrace{\mathrm{e}^{-0\cdot t}}_{=1}\frac{\sin(t)}{t}\,\mathrm{d}t \;= \mathcal{L}\Big(\frac{\sin(t)}{t}\Big)(0) \;\overset{\text{by (5b)}}{=\!=}\; \int_0^\infty \widehat{\sin}\,.$$

This, together with the previous line, gives

7c: $$\int_0^\infty \frac{\sin(t)}{t}\,\mathrm{d}t \quad=\quad \frac{\pi}{2}\,,$$

in an appropriate sense. [The appropriate sense is interpreting LhS(7c) as $\lim_{U\nearrow\infty}\int_0^U \frac{\sin(t)}{t}\,\mathrm{d}t$.]

Preliminaries

The ***Heaviside fnc*** $\mathbf{H}:\mathbb{R}\to\{0,1\}$ is 0 on $(-\infty, 0)$ and is 1 on $[0,+\infty)$. Thus[♡2]

$$\sum_{K=1}^{\infty} \mathbf{H}(x-K) \;=\; \lfloor x \rfloor\,, \qquad \text{for each } x \geqslant 0.$$

Let $\boldsymbol{\delta}$ denote the ***Dirac delta "function"***.[♡3] Write $\boldsymbol{\delta}_5$ for its translate $x \mapsto \boldsymbol{\delta}(x-5)$. So for P a posreal, and f *continuous* at P:

$$\int_0^\infty f \cdot \boldsymbol{\delta}_{\mathrm{P}} \;=\; f(\mathrm{P})\,.$$

In particular, for each f which is continuous:

8a: $$[f \circledast \boldsymbol{\delta}_5](t) \;=\; \left\{\begin{matrix} 0 & \text{, if } t \in [0,5) \\ f(t-5) & \text{, if } t \in [5,\infty) \end{matrix}\right\}.$$

I.e. $\quad f \circledast \boldsymbol{\delta}_5 \;=\; \mathbf{T}_5\big(\mathbf{H}\cdot f\big)$.

Periodicity. A posreal number P is a "***period*** of fnc f" if

$$\forall x: \quad f(x+\mathrm{P}) \;=\; f(x)\,.$$

Typically, there is a *smallest* such period, which is called the "***least-period*** of f".

For a posreal P, let $f_{\langle \mathrm{P}\rangle} := f\cdot\mathbf{1}_{[0,\mathrm{P})}$ abbreviate what I will call "f clipped at P". That is, we restrict f to interval $[0,\mathrm{P})$. Thus

$$\widehat{f_{\langle \mathrm{P}\rangle}}(s) \;=\; \int_0^{\mathrm{P}} \mathrm{e}^{-st} f(t)\,\mathrm{d}t\,.$$

8b: Periodicity Theorem. *Suppose $f, g \in \mathrm{Ord}(\boldsymbol{\mu})$ and P is a posreal. Then*

1: $\mathcal{L}\Big(\mathbf{H}(t-\mathrm{P})\cdot f(t-\mathrm{P})\Big)(s) = \mathrm{e}^{-s\mathrm{P}} \cdot \widehat{f}(s)$.

2: $\mathcal{L}\Big(\mathbf{H}(t-\mathrm{P})g(t)\Big)(s) = \mathrm{e}^{-s\mathrm{P}} \cdot \mathcal{L}\Big(g(t+\mathrm{P})\Big)(s)$.

3: *Suppose now that P is a period of f. Then*

$$\widehat{f}(s) \;=\; \widehat{f_{\langle \mathrm{P}\rangle}}(s) \Big/ [1-\mathrm{e}^{-s\mathrm{P}}]\,. \qquad \diamondsuit$$

Proof of (8b.1). The LhS equals $\int_{\mathrm{P}}^\infty \mathrm{e}^{-st} f(t-\mathrm{P})\,\mathrm{d}t$. CoV $x = t-\mathrm{P}$ gives $\mathrm{d}x = \mathrm{d}t$, and thus

$$\mathrm{LhS}(8b.1) = \int_0^\infty \mathrm{e}^{-s\,[x+\mathrm{P}]} f(x)\,\mathrm{d}x \stackrel{\text{note}}{=\!=} \mathrm{RhS}(8b.1)\,. \blacklozenge$$

Pf (8b.3). Because our "clipped" $f_{\langle \mathrm{P}\rangle}(t)$ equals $[\mathbf{H}(t) - \mathbf{H}(t-\mathrm{P})]\cdot f(t)$, it follows that

$$*: \quad \begin{aligned} \mathcal{L}(f_{\langle \mathrm{P}\rangle}) &= \mathcal{L}(\mathbf{H}\cdot f) - \mathcal{L}\Big(\mathbf{H}(t-\mathrm{P})\cdot f(t)\Big) \\ &\stackrel{\text{P an } f\text{-period}}{=\!=\!=} \mathcal{L}(\mathbf{H}\cdot f) - \mathcal{L}\Big(\mathbf{H}(t-\mathrm{P})\cdot f(t-\mathrm{P})\Big)\,. \end{aligned}$$

Our (8b.1) says that $\mathcal{L}\Big(\mathbf{H}(t-\mathrm{P})\cdot f(t-\mathrm{P})\Big)(s)$ equals $\mathrm{e}^{-s\mathrm{P}}\cdot\widehat{f}(s)$. And, always, $\mathcal{L}(\mathbf{H}\cdot f) = \mathcal{L}(f)$, since the Laplace integral is over $[0,\infty)$. So we can re-write $(*)$ as $\widehat{f_{\langle \mathrm{P}\rangle}}(s) = [1-\mathrm{e}^{-s\mathrm{P}}]\cdot\widehat{f}(s)$. $\blacklozenge$

[♡2] For a real x, the expression $\lfloor x \rfloor$ is called the ***floor*** of x; it is the largest integer less-equal x. So $\lfloor \boldsymbol{\pi} \rfloor$ is 3.

Use DE to mean '*differential eqn*'.

[♡3] It is actually a *Schwartzian distribution*, named after Laurant Schwartz. As distributions, $\mathbf{H}' = \boldsymbol{\delta}$.

Using periodicity. Function sin() is periodic, with period $2\boldsymbol{\pi}$. So for each posint K:

$$\big[\sin \circledast \boldsymbol{\delta}_{2\boldsymbol{\pi}K}\big](t) \;=\; \begin{cases} 0 & \text{, if } t \in [0, 2\boldsymbol{\pi}K) \\ \sin(t) & \text{, if } t \in [2\boldsymbol{\pi}K, \infty) \end{cases} .$$

For $N \in \{1,2,3,\dots\} \cup \{\infty\}$, define the sum

$$R_N \;:=\; \sum_{K=1}^{N} \boldsymbol{\delta}_{2\boldsymbol{\pi}K} .$$

For N finite, then,

$$\big[\sin \circledast R_N\big](t) \;=\; \Big\lfloor \frac{t}{2\boldsymbol{\pi}} \Big\rfloor \cdot \sin(t)$$

holds when $0 \leqslant t < 2\boldsymbol{\pi}[N{+}1]$. There are no convergence problems as $N \nearrow \infty$. So $\forall t \in [0\,..\,\infty)$:

8c: $$\big[\sin \circledast R_\infty\big](t) \;=\; \Big\lfloor \frac{t}{2\boldsymbol{\pi}} \Big\rfloor \cdot \sin(t) .$$

Moreover, for each finite N:

8c$_N$: $$\big[\sin \circledast R_N\big](t) \;=\; \mathrm{Min}\Big(N\,,\, \lfloor\tfrac{t}{2\boldsymbol{\pi}}\rfloor\Big) \cdot \sin(t).$$

Hammering an undamped spring

(One of the textbooks called this *Soldiers marching in cadence*, but the interpretation is less clear, since soldiers already on the bridge are still marching.

There is some historical evidence: Broughton Suspension Bridge, 1831 and Millennium Bridge, 2000 [TY, Molly Militello]. YouTube Synchronization discusses self-developing synchrony, using the Millennium bridge as one example [TY, Alexander Bush].)

With y the unknown-fnc, let us examine DE

$\mathfrak{D}$: $$y'' + y \;=\; \sum_{K=1}^{\infty} \boldsymbol{\delta}_{2\boldsymbol{\pi}K} .$$

Given two complex numbers α, β, let $(\mathfrak{D}_{\alpha,\beta})$ mean the DE together with initial conditions $y(0) = \alpha$ and $y'(0) = \beta$.

Consider the corresponding ZeroTar[♡4] IVP

$$z'' + z \;=\; 0\,, \qquad \textit{with } z(0) = \alpha \textit{ and } z'(0) = \beta .$$

[♡4] Some textbooks call this (yuck!) "homogeneous".

Its solution is

8d: $$z \;=\; [\alpha \cdot \cos] + [\beta \cdot \sin] .$$

So the soln, y, to $(\mathfrak{D}_{0,0})$ will, when added to z, give the solution to $(\mathfrak{D}_{\alpha,\beta})$.

Courtesy (5c) [Lap-of-Deriv lemma]

$$\begin{aligned} \mathcal{L}\big(y'' + y\big)(s) \;&=\; \Big[s^2 \cdot \widehat{y}(s) \;-\; \overbrace{[s \cdot y(0) + y'(0)]}^{=0}\Big] + \widehat{y}(s) \\ &=\; [s^2 + 1] \cdot \widehat{y}(s) . \end{aligned}$$

Now $(\mathfrak{D})$ says that $\mathcal{L}\big(y'' + y\big) = \mathcal{L}(R_\infty)$. Dividing,

$$\widehat{y}(s) \;=\; \tfrac{1}{s^2+1} \cdot \widehat{R_\infty}(s) .$$

The RhS is a product, so its inv-lap-xform is a convolution.

$$y \;=\; \sin \circledast R_\infty .$$

In other words, the function

8e: $$y_{\alpha,\beta}(t) \;:=\; \alpha\cos(t) + \beta\sin(t) + \Big\lfloor \frac{t}{2\boldsymbol{\pi}} \Big\rfloor \cdot \sin(t)$$

is the general solution to IVP $(\mathfrak{D}_{\alpha,\beta})$.

At this point, it would be good to have a careful sketch of $y_{\alpha,\beta}()$. One could also analyze, at each time t, the potential and kinetic energy in the spring. If the spring is damped, is there a net absorption, or loss, of energy?

Suppose the tapping on the spring was <u>not</u> aligned with the resonant frequency of the spring —would the spring nonetheless absorb energy?

Random hammering. Times $0 < G_1 < G_2 < \dots$ define

$$R_{\mathbf{G}} \;:=\; \sum_{K=1}^{\infty} \boldsymbol{\delta}_{G_K} .$$

The above reasoning applied to DE $\boxed{y'' + y = R_{\mathbf{G}}}$ yields $y = \sin \circledast R_{\mathbf{G}}$, as before. Computing the convolution gives

8f: $$y(t) \;=\; \sum_{K=1}^{\infty} \mathbf{H}(t - G_K) \cdot \sin(t - G_K) .$$

Adding $[\alpha\cos(t) + \beta\sin(t)]$ hands us the general solution $y_{\alpha,\beta}$.

Square Wave

For a period P>0 and a ***dUty cycle*** $U \in [0,1]$, define the ***square-wave fnc*** $\mathrm{SW}_{P,U}()$ to be *the* P-periodic fnc mapping $\mathbb{R}\to\mathbb{R}$, which agrees with $\mathbf{1}_{[0,UP)}$ on interval $[0,P)$; the ***pulse width*** is UP.

So $\mathrm{SW}_{P,0} \equiv 0$ and $\mathrm{SW}_{P,1} \equiv 1$. Thus we expect

$$*: \quad \widehat{\mathrm{SW}_{P,0}} \equiv 0 \quad \textit{and} \quad \widehat{\mathrm{SW}_{P,1}}(s) = 1/s\,.$$

Setting $f := \mathrm{SW}_{P,U}$, our Periodicity Thm says that $\widehat{f_{\langle P\rangle}}(s)$ equals

$$\int_0^P e^{-st}\cdot \mathbf{1}_{[0,UP)}(t)\,dt \overset{\text{since } U\leqslant 1}{=\!=\!=} \int_0^{UP} e^{-st}\,dt = \frac{1}{s}\cdot[1 - e^{-s\cdot UP}]\,.$$

Consequently,

$$†1: \quad \widehat{\mathrm{SW}_{P,U}}(s) = \frac{1}{s}\cdot M_{P,U}(s)\,,$$

where this multiplier fnc $M_{P,U}$ is

$$†2: \quad M_{P,U}(s) = \frac{1 - e^{-UP\cdot s}}{1 - e^{-P\cdot s}} \overset{\text{note}}{=\!=\!=} \frac{e^{P\cdot s} - e^{FP\cdot s}}{e^{P\cdot s} - 1}\,;$$

here, $F := 1 - U$ represents the "offF" part of the cycle. Happily, (†1, †2) is consistent with $(*)$.

Square-Wave into Spring. Laplace transforming $\boxed{y'' + y = \mathrm{SW}_{P,U}}$ yields

$$¥: \quad [s^2+1]\cdot\widehat{y}(s) = \widehat{\mathrm{SW}_{P,U}}(s)\,.$$

Using initial conditions $y'(0) = 0$ and $y(0) = 0$, the earlier derivation gives

$$9: \quad y = \sin \circledast \mathrm{SW}_{P,U}\,.$$

Alternatively, write (¥) as

$$[s^2+1]\cdot\widehat{y}(s) = \tfrac{1}{s}\cdot M_{P,U}(s)\,. \quad \textit{So}$$

$$\widehat{y}(s) = \frac{1}{s^2+1}\cdot\frac{1}{s}\cdot M_{P,U}(s)\,. \quad \textit{Thus}$$

$$9': \quad y = \sin\circledast\mathbf{1}\circledast\mathcal{L}^{-1}(M_{P,U}) = [1-\cos]\circledast\mathcal{L}^{-1}(M_{P,U})\,.$$

[For pictures]