Lagrange Multipliers

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Cubic-surface Problem

On $\mathbb{R}^3$, find all extreme points $P = \big(x, y, z\big)$ of function

$$\psi(P) \;:=\; x^2 + y^2 + z^2\,,$$

subject to the *two* constraints that

$$C_f\textbf{:} \qquad x\cdot y\cdot z \;=\; 54\,; \quad \text{(a cubic surface)}$$
$$C_g\textbf{:} \qquad x+y+z \;=\; 12\,. \quad \text{(a plane)}$$

A picture. Note that the objective fnc *and* the two constraints are symmetric under all permutations of x, y, z. Thus our set of Lagrange points must have a six-fold ($6 = 3!$) symmetry.

A good picture suggests that the only Lagrange pts have some two of x, y, z equal. In the case that $x = z$, say, a picture suggests that there are 3 solutions; two solns where the common value is positive and one soln with it negative. Thus the $z{=}y$, $y{=}x$ and $x{=}z$ solutions would give us 9 Lagrange-pts all together, but only 3 *types* of Lagrange-pt.

A solution

Define *specifier functions*

$$f(P) \;:=\; x{\cdot}y{\cdot}z \quad\text{and}$$
$$g(P) \;:=\; x+y+z\,,$$

so that locus (C_f) is some level-set of f, and locus (C_g) is some level-set of g.

The Lagrange eqns. Using variables α, β to be the Lagrange multipliers for f and g, respectively, the Lagrange eqns become:

$$L_x\textbf{:} \qquad \psi_x \;=\; \alpha f_x \;+\; \beta g_x\,, \text{ i.e } \quad 2x \;=\; \alpha\cdot yz + \beta\cdot 1\,.$$
$$L_y\textbf{:} \qquad \psi_y \;=\; \alpha f_y \;+\; \beta g_y\,, \text{ i.e } \quad 2y \;=\; \alpha\cdot zx + \beta\cdot 1\,.$$
$$L_z\textbf{:} \qquad \psi_z \;=\; \alpha f_z \;+\; \beta g_z\,, \text{ i.e } \quad 2z \;=\; \alpha\cdot xy + \beta\cdot 1\,.$$

We have 5 unknowns: The spatial variables x, y, z and the Lagr.vars α, β. We have 5 eqns: The constraints eqns (C_f), (C_g), and the Lagr.eqns (L_x), (L_y), (L_z).

Solving this system of eqns. If possible, we try to eliminate the Lagrange-vars first. We can subtract (L_y) from (L_z) to obtain

$$(L_z)\text{-}(L_y)\textbf{:} \qquad 2[z-y] \;=\; \alpha xy - \alpha zx \;\stackrel{\text{note}}{=\!=}\; \alpha x[y-z]\,.$$

We now use symmetry to solve the system.

Case $\boxed{x = y = z}$**.** Equation (C_g) yields that $x = y = z = 4$. So their product, $x\cdot y\cdot z$, is 64. This contradicts (C_f). THUS: *No solutions*, here.

Case $\boxed{x, y, z \text{ are distinct}}$**.** Then $y - z \neq 0$ so we may divide equation $(L_y) - (L_z)$ to conclude that $^-2 = \alpha x$. But $^-2$ is not zero, so α must be non-zero and so $x = \frac{^-2}{\alpha}$.

By symmetry $[z - x \neq 0, \text{ etc}]$, we may conclude that y, too, equals $\frac{^-2}{\alpha}$. Alas, this contradicts that $x \neq y$. SO: *No solutions* in this case.

Case $\boxed{\text{Exactly two of } x, y, z \text{ are equal}}$**.** Use $\mathbf{t}$ for the value which occurs twice among $\{x, y, z\}$, and use $\mathbf{u}$ for the value which occurs uniquely. Eqn (C_g) tells us then that $\mathbf{t}+\mathbf{t}+\mathbf{u} = 12$, i.e, $\mathbf{u} = 12 - 2\mathbf{t}$. From (C_f), then, $54 = \mathbf{t}^2\mathbf{u} \stackrel{\text{note}}{=\!=} \mathbf{t}^2[12 - 2\mathbf{t}]$. Multiplying out, then,

$$0 \;=\; q(\mathbf{t})\,, \quad \text{where } q(t) := t^3 - 6t^2 + 27.$$

Evidently $q(3)$ is zero, so we divide $t - 3$ into $t^3 - 6t^2 + 27$ to obtain this factorization:

$$q(t) \;=\; [t-3][t^2 - 3t + 9]\,.$$

The QF (Quadratic Formula) tells us that the zeros of $t^2 - 3t + 9$ are $\frac{3}{2}[1 \pm \sqrt{5}\,]$. And since $\mathbf{u} = 12 - 2\mathbf{t}$, we obtain this table of Lagrange pairs $(\mathbf{t}, \mathbf{u})$. In the table, the three values of $\mathbf{t}$ decrease from left to right.

	Soln 1	*Soln 2*	*Soln 3*
$\mathbf{t}$	$\frac{3}{2}[1+\sqrt{5}\,]$	3	$\frac{3}{2}[1-\sqrt{5}\,]$
$\mathbf{u}$	$3[3-\sqrt{5}\,]$	6	$3[3+\sqrt{5}\,]$
$\frac{1}{9}\psi(\mathbf{t},\mathbf{t},\mathbf{u})$	$17-5\sqrt{5}$	6	$17+5\sqrt{5}$
Size	*closest*	*med.*	*farthest*

The Method in General

Suppose we have (or have derived) K constraints in N-dim space; use $Q = (x_1, \ldots, x_N)$ for a general point in N-space. We wish to find all constrained extrema of a function $\psi{:}\mathbb{R}^N \to \mathbb{R}$.

Step I. Write down a formula for ψ, the ***objective function***, and label the constraint equations $(C_1),\ldots,(C_K)$.

Step II. Write down definitions ($:=$) of specifier functions $f_1, \ldots, f_K$ so that each (C_j) is a level-set of the corresponding f_j.

Introduce unknowns $\alpha_1, \ldots, \alpha_K$ to be the ***Lagrange multipliers*** of $f_1, \ldots, f_K$.

Step III. For $m = 1, 2, \ldots, N$, write down the ***Lagrange equations***

$$L_m\textbf{:} \qquad \frac{\partial\psi}{\partial x_m} \quad = \quad \sum_{j=1}^{K}\Big[\alpha_j \cdot \tfrac{\partial f_j}{\partial x_m}\Big].$$

We now have a system of $K+N$ eqns $(C_1),\ldots,(C_K)$, $(L_1),\ldots,(L_N)$ in $N+K$ unknowns $x_1, \ldots, x_N, \alpha_1, \ldots, \alpha_K$.

Step IV. A soln to the above SoE (System of Eqns) has form $(\vec{x}, \vec{\alpha})$, where $\vec{x}$ abbreviates $x_1, \ldots, x_N$ and $\vec{\alpha}$ abbreviates $\alpha_1, \ldots, \alpha_K$.

Find all *solutions* to this system. Each tuple $P := (x_1, \ldots, x_N)$ is a ***Lagrange point***.

Step V. Evaluate ψ at each Lagrange pt and find those which give largest and smallest values.

Will we examine the comet?

Our astronomy community has informed us that a comet, in parabolic orbit about the Sun, is going to arrive in about 2 years.

Unfinished.

Problems

LM1: Find all points P on the intersection of the sphere $x^2+y^2+z^2 = 1^2$ and plane $3x+2y = z$ which extremize ψ, where $\psi(Q) := y+z$.

LM2: Consider light rays emitted from point $Q_1 := (0,1)$, bounce once off the $xy{=}1$ hyperbola –let P denote the point where it hits– and then arrive at the origin Q_2.

Carefully perform steps **I, II, III** *en route* to finding all points P which minimize the length of the polygonal path Q_1PQ_2. (Do not solve the system of eqns; just set it up.)

LM3: Set up the above light-ray problem but in 3-space, with $Q_1 := (-79, 0, 0)$ and $Q_2 := (3, 23, 29)$, using an ellipsoidal mirror

$$\frac{x^2}{3^2} + \frac{y^2}{5^2} + \frac{z^2}{7^2} \quad = \quad 1^2\,.$$

(Just set up the SoE; do not solve it.)

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