Class Notes for Probability & Potential Theory: Probability

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Notation. WLOG: '*Without loss of generality*'. TFAE: '*The following are equivalent*'. ITOf: '*In Terms Of*'. OTForm: '*of the form*'. FTSOC: '*For the sake of contradiction*'. Use iff: '*if and only if*'.
IST: '*It Suffices to*' as in ISTShow, ISTExhibit.
Use w.r.t: '*with respect to*' and s.t: '*such that*'.
Latin: e.g: *exempli gratia*, '*for example*'. i.e: *id est*, '*that is*'. N.B: *Nota bene*, '*Note well*'. QED: *quod erat demonstrandum*, meaning "end of proof". r.var, or r.v.: '*random variable*'

Prolegomenon

Our probability space is $\mathbf{\Omega} := (\Omega, \mathcal{F}, \mathsf{P})$. A r.v.seq $\vec{\mathsf{S}}$ is ***Cauchy in probability***, written P-***Cauchy***, if: $\forall\varepsilon, \exists\delta$ *so that for all large j and k:*

$$\dagger: \qquad \mathsf{P}(|\mathsf{S}_j - \mathsf{S}_k| > \varepsilon) \leqslant \delta .$$

1: Prop'n. *A r.v.seq $\vec{\mathsf{S}}$ is P-Cauchy IFF there exists a r.v. Z such that $\mathsf{P}\text{-}\lim(\vec{\mathsf{S}}) = \mathsf{Z}$.* ◊

Proof of (⇒). Take an index-sequence $\vec{N}$ so that

$$\ddagger: \qquad \mathsf{P}(|\mathsf{S}_{N_j} - \mathsf{S}_{N_{j+1}}| > 1/3^j) \leqslant 1/5^j .$$

Since $j \mapsto 1/5^j$ is summable, Borel-Cantelli allows us to delete a nullset so that: $\forall\omega, \forall_{\text{large}} j$: $|\mathsf{S}_{N_j}(\omega) - \mathsf{S}_{N_{j+1}}(\omega)| \leqslant 1/3^j$. Thus

$$\mathsf{Z}(\omega) := \lim_j \mathsf{S}_{N_j}(\omega) \quad \text{exists in } \mathbb{R}.$$

Consequently $\mathsf{S}_{N_j} \xrightarrow{\mathsf{P}} \mathsf{Z}$, as $j \to \infty$. Now (‡) yields $\mathsf{P}\text{-}\lim(\vec{\mathsf{S}}) = \mathsf{Z}$. ♦

Fixing an $\varepsilon \geqslant 0$, notice that

$$|\mathsf{Y}|_\varepsilon := \mathsf{P}(|\mathsf{Y}| \geqslant \varepsilon)$$

is a pseudo-norm.

2: PIMZ Lemma (Pairwise-indep mean-zero). *We have $\vec{\mathsf{X}}$, a pairwise-independent list of mean-zero r.vars. For each N, then,*

$$\mathsf{E}\Big(\Big[\sum_1^N \mathsf{X}_j\Big]^2\Big) = \sum_1^N \mathsf{E}(\mathsf{X}_j{}^2) . \qquad ◊$$

3: Lemma. *$\vec{\mathsf{X}}$ a pairwise-independent process of mean-zero r.vars. Let $\mathsf{S}_k := \sum_{j=1}^k \mathsf{X}_j$ be the partial sums. If $\sum_1^\infty \mathsf{E}(\mathsf{X}_j{}^2) < \infty$, then $\mathsf{P}\text{-}\lim(\vec{\mathsf{S}})$ exists.*
Proof. PIMZ and Chebyshev and (1). ◊

Observe. A sequence $\mathbf{s}$ of reals is Cauchy if $\text{Diam}(\mathbf{s}_{(A..\infty)}) \to 0$ as $A \to \infty$. So if $\mathbf{s}$ not Cauchy then $\exists\varepsilon$ with the diameter always strictly exceeding 2ε. Hence $\forall A, \exists j > A$ with $|s_j - s_A| \geqslant \varepsilon$.

Apply this to a r.v.seq $\vec{\mathsf{S}}$ which *fails* to a.s.-converge. There is then an ε so that $\bigcap_{A=1}^\infty G_A$ has positive probability; say, exceeding a particular $\delta{>}0$. Here

$$G_A := \{ \sup_{j \in (A..\infty)} |\mathsf{S}_j - \mathsf{S}_A| \geqslant \varepsilon \} .$$

For each A, then, $\mathsf{P}(G_A) > \delta$. Fixing A, let $H^B := \{\sup_{j\in(A..B]} |\mathsf{S}_j - \mathsf{S}_A| \geqslant \varepsilon\}$ and observe that these sets $H^B \nearrow G_A$ as $B \nearrow \infty$. □

The above yields the following useful condition.

4: Proposition. *Suppose a r.v.seq $\vec{\mathsf{S}}$ fails to a.e.-converge. Then there exist $\varepsilon, \delta > 0$ so that*

$$\forall A, \exists B \in (A..\infty) \text{ for which} \quad \mathsf{P}\Big(\sup_{j\in(A..B]} |\mathsf{S}_j - \mathsf{S}_A| \geqslant \varepsilon\Big) > \delta . \qquad ◊$$

5: Skorokhod sup Lemma. (*Breiman: 3.21 P.45*) *With $\vec{\mathsf{Y}}$ an independent process, let $\mathsf{S}_k := \sum_{j=1}^k \mathsf{Y}_j$ be the partial sums. For each posreal ε, and each N:*

$$\mathsf{P}\Big(\sup_{j\in[1..N]} |\mathsf{S}_j| \geqslant 2\varepsilon\Big) \leqslant \frac{\mathsf{P}\big(|\mathsf{S}_N| > \varepsilon\big)}{1 - \mathbf{c}} ,$$

as long as the number

$$\mathbf{c} := \sup_{j\in[1..N]} \mathsf{P}(|\mathsf{S}_N - \mathsf{S}_j| \geqslant \varepsilon)$$

is strictly less than 1. ◊

6: *Remark.* Translating where we start the partial sums gives the following. For all natnums $A < B$:

$$\mathsf{P}\big(\sup_{j\in(A\,..\,B]} |\mathsf{S}_j - \mathsf{S}_A| \geqslant 2\varepsilon\big) \;\leqslant\; \frac{\mathsf{P}\Big(|\mathsf{S}_B - \mathsf{S}_A| \geqslant \varepsilon\Big)}{1 - \sup_{j\in(A\,..\,B]} \mathsf{P}(|\mathsf{S}_B - \mathsf{S}_j| \geqslant \varepsilon)}\,.$$

(Well, as long as the denominator supremum is less than 1.) Suppose $\vec{\mathsf{S}}$ is P-Cauchy. As $A, B \to \infty$, then, the RhS tends to $\frac{0}{1} = 0$. This together with (4) the following nifty corollary. □

7: Theorem. *Suppose $\vec{\mathsf{S}}$ is the process of partial-sums of an independent process. If $\vec{\mathsf{S}}$ converges in probability to a r.var Z, then $\vec{\mathsf{S}}$ converges **almost surely** to Z.* ♢

Proof of Skorokhod. Fix $\varepsilon, N, \mathbf{c}$ from (5) and let

$$V \;\;:=\;\; \Big\{\sup_{j\in[1\,..\,N]} |\mathsf{S}_j| \;\geqslant\; 2\varepsilon\Big\}\,.$$

(V for Very big.) Define a **stopping time** $\tau\colon V \to [1\,..\,N]$ by letting $\tau(j)$ be *smallest* such that $|\mathsf{S}_j| \geqslant 2\varepsilon$. On $\Omega \smallsetminus V$ let $\tau := +\infty$.

Consider a $j \in [1\,..\,N]$; say $j = 5$. The event $\{|\mathsf{S}_N - \mathsf{S}_5| \leqslant \varepsilon\}$ lies in $\text{Fld}(\mathsf{Y}_6, \ldots, \mathsf{Y}_N)$, whereas event $\{\tau() = 5\} \in \text{Fld}(\mathsf{Y}_1, \ldots, \mathsf{Y}_5)$. So these two events are independent. Thus

$$\mathsf{P}\begin{pmatrix}|\mathsf{S}_N - \mathsf{S}_5| \leqslant \varepsilon \\ \text{and } \tau() = 5\end{pmatrix} \;=\; \mathsf{P}\big(|\mathsf{S}_N - \mathsf{S}_5| \leqslant \varepsilon\big)\cdot \mathsf{P}(\tau = 5)$$
$$\geqslant\; [1 - \mathbf{c}]\cdot \mathsf{P}(\tau = 5)\,.$$

Now note that $|\mathsf{S}_5| \geqslant 2\varepsilon$ <u>and</u> $|\mathsf{S}_N - \mathsf{S}_5| \leqslant \varepsilon$ together imply that $|\mathsf{S}_N| \geqslant \varepsilon$. Consequently

$$\mathsf{P}\begin{pmatrix}|\mathsf{S}_N| \geqslant \varepsilon \\ \text{and } \tau() = 5\end{pmatrix} \;\geqslant\; [1 - \mathbf{c}]\cdot \mathsf{P}(\tau = 5)\,.$$

Replace "5" by "j" and sum, to conclude that

$$\mathsf{P}\Big(\{|\mathsf{S}_N| \geqslant \varepsilon\} \cap V\Big) \;=\; \sum_{j\in[1\,..\,N]} \mathsf{P}\begin{pmatrix}|\mathsf{S}_N| \geqslant \varepsilon \\ \text{and } \tau() = j\end{pmatrix}$$
$$\geqslant\; [1 - \mathbf{c}]\cdot \sum_{j\in[1\,..\,N]} \mathsf{P}\Big(\tau() = j\Big)$$
$$=\; [1 - \mathbf{c}]\cdot\; \mathsf{P}(V)\,.$$

Consequently $\mathsf{P}\Big(|\mathsf{S}_N| \geqslant \varepsilon\Big) \;\geqslant\; [1 - \mathbf{c}]\cdot\; \mathsf{P}(V)$, as desired. ♦

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