Gambling

Let's play BLACKJACK with these cards:

J♥ T♥ 9♥ 8♥ 7♥ 6♥ 5♥ 4♥ 3♥

where J♥ =11 *and* T♥ =10;

Blackjack(= 21) is the goal.

Abby and Bert alternate taking a card from the list and putting the card in their hand.

After 5 turns, perhaps the position is this:

J♥ 9♥ 5♥ 4♥

Abby's hand	**Bert's** hand
T♥ 3♥ 7♥	6♥ 8♥ ?

A player wins if, after adjoining a card to his hand, he now has some three cards summing to *Blackjack*. If *all* the cards are in player's hands, yet nobody has won, then the game is drawn.

Have You played this game?

Thinking Isomorphically. Suppose we subtract 1 from each card-value. We're now playing with this deck:

T♥ 9♥ 8♥ 7♥ 6♥ 5♥ 4♥ 3♥ 2♥

Since we need 3 cards to win, our new target is $21-3$, i.e *18*. This gives us a game which is isomorphic to BLACKJACK.

What happens if we follow this line of thought to its logical conclusion ?

Logical Conclusion. ... following that line of thought, we iterate the idea. Since $\frac{21}{3} = 7$, subtracting 7 from each card gives symmetry

+4♥ +3♥ +2♥ +1♥ 0♥ −1♥ −2♥ −3♥ −4♥

with *0*, now, as the target sum.

But... so what ?

Ringing the Bell. The symmetry suggests something ***Magic!*** that we've likely seen before...

−3	**+4**	**−1**
+2	**0**	**−2**
+1	**−4**	**+3**

All eight TTTs (tic-tac-toes) [three vertical, three horizontal, and two diagonal] sum to *0*. And the remaining $\binom{9}{3} - 8 = 84 - 8 = 76$ '***bad***' (non-TTT) triples, do *not* sum to *zero*. Thus:

∗: BLACKJACK *is game-isomorphic to* TTT.

[Exer: Even though there are 76 bad triples, why do we only need to check 2 of them?]

Soln is temporarily hidden.

Automorphisms. 3×3-TTT has 8 ***automorphisms*** [self-isomorphisms]. One can check that the only permutations of the 9 cells that preserve TTTs, are the geometric symmetries of a square. This group of 8 is called the *4^{th}-**dihedral** group*. It comprises the 4 rotations of the board, together with flipping the board over, then rotating.

[Exer: What is the TTT-automorphism-group of the 4×4 board? (Shhh..., –are there are some *non-geometric* autos?)]

We can carry these automorphisms back to BLACKJACK, thusly:

4♥	J♥	6♥
9♥	7♥	5♥
8♥	3♥	T♥

BLACKJACK *is suddenly easier to win.*
(Las Vegas, *here I come!*)