Intermediate-value Theorem

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Bernard Bolzano (1781–1848) proved the following form of the Intermediate-value Theorem.

1: IVT. *Suppose $f\colon [a,b] \to \mathbb{R}$ is continuous, with $f(a)$ and $f(b)$ non-zero and having different signs. Then there exists a point $c \in (a,b)$ which is a zero of f, i.e, $f(c) = 0$.* ◊

Proof. WLOGenerality, $f(a) < 0$ and $f(b) > 0$; otherwise, simply replace f by $-f$ (which preserves continuity) and note that a zero of $-f$ is a zero of f.

Let $L_0 := a$ and $R_0 := b$. For stage $n = 1, 2, \ldots$, either up to some integer K, or out to ∞, I will produce numbers L_n and R_n such that:

i[n]: $a \leqslant L_{n-1} \leqslant L_n < R_n \leqslant R_{n-1} \leqslant b$;

ii[n]: $R_n - L_n = \frac{1}{2}[R_{n-1} - L_{n-1}]$;

iii[n]: $f(L_n) < 0 < f(R_n)$.

Stage-n construction. Let M be the midpoint of interval $[L_{n-1}, R_{n-1}]$, i.e, $M := \frac{1}{2}[L_{n-1} + R_{n-1}]$.

$\boxed{\text{Case: If } f(M) \text{ is zero, then STOP}}$ Set $K := n-1$. By (i[K]), note that M is strictly between a and b. So $c := M$ fulfills the conclusion of the theorem.

$\boxed{\text{Case: Otherwise, } f(M) \neq 0.}$ *If $f(M)$ negative* then let $L_n := M$ & $R_n := R_{n-1}$. *If $f(M)$ positive* then let $L_n := L_{n-1}$ & $R_n := M$. In either case, conditions (i,ii,iii[n]), automatically hold.

Last step. WLOGenerality, we may assume that our construction never STOPped. So we have two sequences, $\vec{L} := (L_n)_{n=0}^{\infty}$ and $\vec{R} := (R_n)_{n=0}^{\infty}$.

By (i), $\vec{L}$ is increasing and is bounded above by b. Since a bounded monotone seq must converge, $L_\infty := \lim_{n\to\infty} L_n$ exists; it is in interval $[a,b]$, courtesy (i).

Thus f is defined –hence continuous– at L_∞, so $f(L_\infty)$ equals $\lim_n f(L_n)$. And $f(L_\infty) \overset{\text{must}}{\leqslant} 0$ since each $f(L_n) \leqslant 0$.

Analogously, $f(R_\infty) := \lim_{n\to\infty} f(R_n)$ exists, and is non-negative. Furthermore

$$\begin{aligned} R_\infty - L_\infty &= \lim_{n\to\infty} [R_n - L_n]\,, \quad \text{by } \textit{what} \text{ thm?}, \\ &= \lim_{n\to\infty} [\tfrac{1}{2}]^n \cdot [b-a]\,, \quad \text{by (iii) and induction}, \\ &= 0\,. \end{aligned}$$

Thus R_∞ and L_∞ equal a common value, call it c, in interval $[a,b]$. The preceding paragraphs tell us that $f(c) \leqslant 0$ *and* $f(c) \geqslant 0$; so $f(c)$ must be zero. Hence $c \notin \{a,b\}$. ♦

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