DynSys MTG 6401

Ergodic HW téssera

Prof. JLF King 31Aug2015

(*Due Wednesday, 07Oct. Please staple this sheet as the first page of your write-up.*)

Prolegomenon. To give you guys a break, here is a undergraduate combinatorics problem. (I'll connect it to Ergodic Theory in the ADDENDUM.)

Let $\Delta(z) \in [1 \mathinner{..} 9]$ denote the *high-order digit* of the base-ten numeral for $z \in \mathbb{R}$. So $\Delta(-\boldsymbol{\pi}/100)$ is 3. (Oh, we have a measure-zero set to fret about. For dyadic rationals $z \neq 0$, use the decimal expansion which is eventually the digits *000* $\cdots$ forevermore. As for $z=0$, define $\Delta(0)$ as you see fit.)

Below, let $\log()$ denote $\log_{10}()$, and let $\mathbb{K}$ be the unit circle.

H8: For each digit $d \in \{1, 2, \ldots, 9\}$ let U_d and L_d be the upper and lower densities of the set

$$E_d \quad := \quad \{n \in \mathbb{Z}_+ \mid \Delta(n) = d\}.$$

In HW3, I sketched an argument showing that $L_1 = \frac{1}{9}$ and $U_1 = \frac{5}{9}$. Please compute the other densities. Which values of d make the ratio U_d/L_d an integer?

ADDENDUM. In HW1, you showed that the $\times 2$ doubling map on $(\mathbb{R}_+, \text{Leb})$ gave actual densities (i.e, UpperDen=LowerDen) to sets $\{E_d\}_{d=1}^{9}$.

You did this by producing a non-singular semi-conjugacy from $(\times 2\colon \mathbb{R}_+, \text{Leb})$ to rotation $(R_{\boldsymbol{\alpha}}\colon \mathbb{K}, \text{Arclen})$, where $\boldsymbol{\alpha} := \log_{10}(2)$ is the rotation number. So the Birkhoff Ergodic Thm implied[♡1] that $\text{Den}(E_d)$ equals $\log(\frac{d+1}{d})$.

We can interpret (H8) as saying something about an orbit of the map $T(x) := x+1$ on $\mathbb{R}$ relative to the partition $\mathsf{P} := (A_1, \ldots, A_9)$, where A_d is the set of $z \in \mathbb{R}$ with $\Delta(z) = d$.

From (H8), it is easy to see that T-orbit of **every** $z \in \mathbb{R}$ has different upper and lower frequencies of hitting A_1 (indeed, for each of the A_d sets).

Thus there does *not exist* a factor map

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{T} & \mathbb{R} \\ {\scriptstyle f}\big\downarrow & & {\scriptstyle f}\big\downarrow \\ X & \xrightarrow{S} & X \end{array}$$

to an mpt $(S\colon X, \mathcal{X}, \mu)$ on a probability space, where X has a (measurable) ptn $\mathsf{Q} := (B_1, \ldots, B_9)$, with $f^{-1}(B_d) = A_d$ for each d. □

[♡1] Well, *actually* the BirkErgThm implied that for a.e $z \in \mathbb{R}_+$, the density of exponents n st. $[\Delta(z \cdot 2^n) = d]$ is $\log(\frac{d+1}{d})$. However, the ideas around *unique ergodicity* (discussed when we get to §4.7 in our text) will show that in fact the density result holds for *every* z. The "bad" nullset is, in fact, empty.