Ergodic Theory
MTG 6401

Ergodic HW Trois

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(*Due Wednesday, 30Sep2009. Please* ***staple this sheet*** *as the* ***first page*** *of your write-up.*)

Notation. Our prob. space $\mathbb{X} := (X, \mathcal{X}, \mu)$ engenders a *complex* Hilbert space $\mathbf{H} := \mathbb{L}^2(\mu, \mathbb{C})$.

H6: With T is a bi-mpt on $\mathbb{X}$, define $\mathsf{U}{=}\mathsf{U}_T$ by $\mathsf{U}f := f \circ T$, a unitary op on $\mathbf{H}$.

a) For each convergence notion Reg/AbsCes/Cesàro of a sequence of numbers, prove that if

$$*: \qquad \langle f, \mathsf{U}^n g\rangle \underset{n\to\infty}{\longrightarrow} \langle f, \mathbf{1}\rangle \cdot \langle \mathbf{1}, g\rangle$$

for all fncs f,g in some dense subset of $\mathbf{H}$, then it holds for *every* pair f,$g \in \mathbf{H}$.

b) Prove that T is *mixing/weak-mixing/ergodic* as $\forall f,g \in \mathbf{H}$:

$$**: \qquad \langle f, \mathsf{U}^n g\rangle \underset{n\to\infty}{\longrightarrow} \langle f, \mathbf{1}\rangle \cdot \langle \mathbf{1}, g\rangle$$

in the Regular/AbsCesàro/Cesàro sense. Our defn/conclusion-of-thm, for these three properties, is that $(**)$ holds when f and g are indicator-fncs of sets.

H7: Here, $Z_1, Z_2, Z_3, \ldots \subset \mathbb{N}$ are zero-density sets.

i) Prove that each *finite* union $U_K := Z_1 \cup \ldots \cup Z_K$ has zero-density.

ii) For sets Y,$Z \subset \mathbb{N}$, say that **"set Y *eventually includes* Z"** if there exists $n \in \mathbb{N}$ with

$$Y \supset Z \cap [n \,..\, \infty).$$

Construct a *zero-density* Y such that for every index j, our Y eventually-includes Z_j.

APPENDIX. For a sequence $\vec{\mathbf{c}} := (c_0, c_1, \ldots)$, write $c_n \xrightarrow[n\to\infty]{\text{Reg}} 0$ for "regular convergence".

Use $c_n \xrightarrow[n\to\infty]{\text{AbsCes}} 0$ for ***absolute Cesàro*** convergence: $\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} |c_k| = 0$.

Use $c_n \xrightarrow[n\to\infty]{\text{Cesàro}} 0$ for plain ***Cesàro*** convergence: $\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} c_k = 0$.

More generally, for a complex number α, say

$$c_n \xrightarrow[n\to\infty]{\text{Reg/AbsCes/Ces}} \alpha$$

IFF $[c_n - \alpha] \xrightarrow[n\to\infty]{\text{Reg/AbsCes/Ces}} 0$, respectively. □

APPENDIX. The ***upper density*** of a subset $Y \subset \mathbb{N}$ is

$$\overline{\text{Den}}(Y) \quad := \quad \limsup_{n\to\infty} \frac{|Y \cap [1 \,..\, n]|}{n}.$$

Its ***lower density*** $\underline{\text{Den}}(Y)$ is defined by $\liminf_{n\to\infty}$. If the two are equal, then Y has a ***density***, written $\text{Den}(Y)$.

So a set with $\overline{\text{Den}} = 0$ has density-zero, and a set with $\underline{\text{Den}} = 1$ has ***full density***, i.e density-one.

Example. Let E be the set of n whose high-order digit is 1. Lower density occurs at numerals OTForm $\overbrace{9\cdots9}^{k}$. Each block of numerals-that-start-with-1 is one of nine equal-length blocks (starting with "1", "2",…,"9"), so $\underline{\text{Den}}(E) = \frac{1}{9}$.

Upper density occurs at $1\overbrace{9\cdots9}^{k}$, but $2\overbrace{0\cdots0}^{k}$, suffices for asymptotics. With $n := 2 \cdot 10^k$, then,

$$|E \cap [1 \,..\, n]| \quad = \quad \tfrac{1}{9}\cdot[10^k - 1] + 10^k.$$

Dividing by n, then sending $k \to \infty$, gives $\overline{\text{Den}}(E) = \frac{10}{18} = \frac{5}{9}$. So E is a set whose upper-density is five times its lower. □