Ergodic Theory
MTG 6401

Ergodic HW Uno

Prof. JLF King
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(*Due Wedn., 16Sept, at the beginning of class. Please* ***staple this sheet*** *as the* ***first page*** *of your write-up.*)

Notation. Let $\mathbb{K}$ be the unit circle, $[0,1)$ wrapped into a circle, with $m()$ its arclength measure. For $\boldsymbol{\alpha} \in \mathbb{R}$, let $R_{\boldsymbol{\alpha}}$ be rotation by $\boldsymbol{\alpha}$ on $\mathbb{K}$.

For explanation of your solutions to each of these three problems, pictures are appropriate.

H1: Here $(T{:}X, \mathcal{X}, \mu)$ is a mpt, and A is an invariant set, i.e, $T^{-1}(A) \overset{\text{a.e}}{=} A$.

a Suppose T is bi-mpt. Construct a set $E \overset{\text{a.e}}{=} A$ so that E is ***exactly-invariant***, i.e, $T^{-1}(E) = E$.

b No longer assume that T is invertible.

With $B := \bigcap_{k=0}^{\infty} T^{-k}(A)$, prove that $B \overset{\text{a.e}}{=} A$ and $T^{-1}(B) \supset B$. Now use B to construct a set $E \overset{\text{a.e}}{=} B$ which is exactly-invariant.

H2: Let $R{=}R_\alpha$ be an irrational rotation on $\mathbb{K}$. Partition the circle into two half-open intervals, say

$$A := \left[0, \tfrac{1}{3}\right) \quad \text{and} \quad B := \left[\tfrac{1}{3}, 1\right).$$

Given a point $z \in \mathbb{K}$, let the ***forward name*** of z be the sequence $(s_0, s_1, s_2, \ldots)$, where s_n is the symbol "A" or "B" as $R^n(z)$ is in A or B.

i Use the density of forward (or reverse) orbits to prove that distinct points have distinct forward names. What does this say about predictability?

ii Describe an algorithm that successively takes in "A" "B" " letters and makes better-and-better guesses as to what point z has a forward-orbit with this half-infinite name.

For each n, having seen the first n letters of the forward-orbit, your algorithm should guess a point x_n in the circle. These guesses should have that property that

① If the name really comes from the forward-orbit of a point $z \in \mathbb{K}$, then the $(x_n)_{n=1}^{\infty}$ sequence converges to z, in the arclength metric.

② If we are being fed a phony `AB` sequence, then at some finite stage our algorithm will cry out **Liar!**

H3: Let $\Delta(z) \in [1\,..\,9]$ denote the *high-order digit* of the base-ten numeral[♡1] for z. So $\Delta(\frac{\pi}{100})$ is 3.

Let $S{:}\mathbb{R}_+\circlearrowleft$ be the ***doubling map***, $z \mapsto 2z$. Produce, with proof, a number $L \in \mathbb{R}_+$ so that, for a.e z

$$\text{1:} \qquad \frac{1}{N}\sum_{k=1}^{N} \Delta(S^k z) \quad \overset{N\to\infty}{\longrightarrow} \quad L\,,$$

using the following method.

Define a specific irr.number $\boldsymbol{\alpha}$ and non-singular map[♡2] $f{:}\mathbb{R}_+ \to \mathbb{K}$ so that this diagram commutes

$$\begin{array}{ccc} \mathbb{R}_+ & \xrightarrow{S} & \mathbb{R}_+ \\ f\big\downarrow & & f\big\downarrow \\ \mathbb{K} & \xrightarrow{R_{\boldsymbol{\alpha}}} & \mathbb{K} \end{array}$$

Prove that your $\boldsymbol{\alpha}$ is irrational.

Now apply (you may use without proof that irr-rotations are ergodic) the Birkhoff Ergodic Thm to **?What?** function, in order to compute L. What can you say/prove about the (ir)rationality of L?

Where is non-singularity of f used in answering the original question?

[♡1] Are there posreals with *multiple* base-ten numerals? How are you going to define $\Delta()$ on these numbers, and how does it affect your result?

[♡2] In the topological category, this f is a semi-conjugacy. Indeed, it is a covering-map in the sense of algebraic topology.