Ergodic HW Efes

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W1: Let $\mathbb{K} \subset \mathbb{C}$ be the unit-circle in the complex plane, equipped with normalized arclength measure. Let $S{:}\mathbb{K}{\to}\mathbb{K}$ be the squaring-map $z \mapsto z^2$, which is a measure-preserving 2-to-1 map on $\mathbb{K}$.

Prove that S is measure-theoretically isomorphic to the 1-sided shift on head-tails coin-flip space, where

$$\begin{aligned} \text{Prob}(\text{Heads}) &= 1/2 \quad \text{and} \\ \text{Prob}(\text{Tails}) &= 1/2\,. \end{aligned}$$

Of course, S is not *topologically* isomorphic to the shift, since S lives on a circle, and the shift lives on a Cantor-set.

W2: For B a complex number, let $\mathcal{N}_B$ be the Newton's method map to find a zero of polynomial

$$\begin{aligned} g_B(x) &:= x^2 - B^2\,. \qquad \text{Thus} \\ \mathcal{N}_B(x) &= [x^2 + B^2]/[2x]\,. \end{aligned}$$

Use $\boldsymbol{\mathfrak{N}}()$ as a shorthand for $\mathcal{N}_1()$. Since $\boldsymbol{\mathfrak{N}}(z)$ equals $\frac{1}{2}[z + \frac{1}{z}]$, it averages a number and its reciprocal.

a Argue that $\mathcal{N}_B$ is well-defined as a map of the Riemann-sphere, $\dot{\mathbb{C}}$, to itself.

b For $B \neq 0$, show that $\mathcal{N}_B$ is *topologically conjugate* to $\mathcal{N}_1$. That is, find a homeomorphism $f{:}\dot{\mathbb{C}}{\to}\dot{\mathbb{C}}$ such that this diagram commutes:

1:
$$\begin{array}{ccc} \dot{\mathbb{C}} & \xrightarrow{\mathcal{N}_B} & \dot{\mathbb{C}} \\ f\big\downarrow & & \big\downarrow f \\ \dot{\mathbb{C}} & \xrightarrow{\mathcal{N}_1} & \dot{\mathbb{C}} \end{array}$$

Thus $\mathcal{N}_B$ and $\mathcal{N}_1$ have "identical dynamics".

Is $\mathcal{N}_0$ well-defined on $\dot{\mathbb{C}}$? Is it conjugate to $\mathcal{N}_1$?

c Find a topological conjugacy, h, which carries $\boldsymbol{\mathfrak{N}}$ to the squaring-map $z \mapsto z^2$ on $\dot{\mathbb{C}}$.

Use symbol $\dot{\mathbb{I}} := [\text{Imaginary axis}] \cup \{\infty\}$, for the "imaginary circle" in the Riemann-sphere. Show that the restriction $\boldsymbol{\mathfrak{N}}\rfloor_{\dot{\mathbb{I}}}$ is conjugate to the squaring-map S on the circle $\mathbb{K}$. Hence $\boldsymbol{\mathfrak{N}}\rfloor_{\dot{\mathbb{I}}}$ is chaotic. Call a point $z \in \dot{\mathbb{I}}$ **"**bad**"** if its forward $\boldsymbol{\mathfrak{N}}$-orbit is dense in $\dot{\mathbb{I}}$. Then the isomorphism of (W1) tells us that there is a dense set of bad points in $\dot{\mathbb{I}}$.

d Use the above h to show that, under $\boldsymbol{\mathfrak{N}}$: The Basin-of-Attraction of fixed-point -1 is $\{z \mid \text{Re}(z) < 0\}$, the open left half-plane. And the $\boldsymbol{\mathfrak{N}}$-BoA of +1 is the open right half-plane.

Addendum. Here are some comments on (W2).

W2b: Define $f(z) = f_B(z) := z/B$. So

2:
$$f^{-1}\Big(\boldsymbol{\mathfrak{N}}\big(f(z)\big)\Big) \;=\; B \cdot \frac{[\frac{z}{B}]^2 + 1^2}{2 \cdot \frac{z}{B}}\,,$$

which indeed equals $\mathcal{N}_B(z)$.

Certainly $\mathcal{N}_0 \stackrel{\text{note}}{=\!=} [x \mapsto \frac{x}{2}]$ is well-defined, but diagram (**??**) fails, since f_B needs $B{\neq}0$ in order to be invertible. To see that there is no *other* topological conjugacy, note:

> *On the Riemann-sphere, the set of fixed-points of $\mathcal{N}_B$ is precisely $\{B, \text{-}B, \infty\}$.*

So $\mathcal{N}_0$ has only *two* fixed-points, but each other $\mathcal{N}_B$ has *three* fixed-points. □

W2c: We do both parts in one swell foop.

On $\dot{\mathbb{I}}$: Our $\boldsymbol{\mathfrak{N}}$ fixes ∞, and sends $\boldsymbol{i} \mapsto 0 \mapsto \infty$.

On $\mathbb{K}$: Our S fixes 1, and sends $\boldsymbol{i} \mapsto \text{-}1 \mapsto 1$.

So we consider the Möbius transformation

3:
$$\begin{array}{c} \infty \mapsto 1 \\ 0 \mapsto \text{-}1 \\ \boldsymbol{i} \mapsto \boldsymbol{i} \end{array} \quad \text{by} \quad y = h(x) := \frac{x-1}{x+1}\,.$$

Its inverse-fnc is $h^{-1}(y) = \frac{1+y}{1-y}$. Composing gives

4:
$$h\Big(\boldsymbol{\mathfrak{N}}\big(h^{-1}(z)\big)\Big) \stackrel{\text{note}}{=\!=} z^2, \text{ i.e } \quad \begin{array}{ccc} \dot{\mathbb{C}} & \xrightarrow{\boldsymbol{\mathfrak{N}}} & \dot{\mathbb{C}} \\ h\big\downarrow & & \big\downarrow h \\ \dot{\mathbb{C}} & \xrightarrow{z \mapsto z^2} & \dot{\mathbb{C}} \end{array}$$

Composing (**??**) with (**??**) carries $\mathcal{N}_B$ to squaring. □

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