Geometry

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ABSTRACT: Heron's theorem on the area of a triangle.
Maximum area of an articulated polygon.
Edgar's way to get an equation of a circle.

The Triangle

Let $\mathbf{T}$ be the triangle $\triangle ABC$. Following the usual convention, $\measuredangle A$ or just A itself will also denote the interior angle at vertex A. The edge opposite vertex A is lowercase $\boldsymbol{a}$, etc. Also, $\boldsymbol{a}$ denotes the *length* of edge $\boldsymbol{a}$.

Tools. Recall the Law of Cosines, which asserts that

LoCos: $$\boldsymbol{c}^2 = \boldsymbol{a}^2 + \boldsymbol{b}^2 - [2\boldsymbol{ab}\cdot\cos(C)].$$

Also note that

1: $$\text{Area}(\triangle ABC) = \tfrac{1}{2}\boldsymbol{ab}\cdot\sin(C),$$

since $\boldsymbol{b}\cdot\sin(C)$ is Len(A-altitude); ie., down to edge $\boldsymbol{a}$.

2: Heron's formula. *Fix $\mathbf{T} := \triangle ABC$. Then*

2i: $$\begin{aligned}&[4\cdot\text{Area}(\mathbf{T})]^2\\ &= [\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}][-\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}][\boldsymbol{a}-\boldsymbol{b}+\boldsymbol{c}][\boldsymbol{a}+\boldsymbol{b}-\boldsymbol{c}]\\ &\overset{\text{note}}{=} 2\Big[\boldsymbol{a}^2\boldsymbol{b}^2+\boldsymbol{b}^2\boldsymbol{c}^2+\boldsymbol{a}^2\boldsymbol{c}^2\Big] - \Big[\boldsymbol{a}^4+\boldsymbol{b}^4-\boldsymbol{c}^4\Big].\end{aligned}$$

Equivalently (and classically),

2ii: $$\text{Area}(\mathbf{T}) = \sqrt{\sigma\cdot[\sigma-\boldsymbol{a}][\sigma-\boldsymbol{b}][\sigma-\boldsymbol{c}]},$$

where $\sigma := \frac{\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}}{2}$ is the ***semi-perimeter*** *of* $\mathbf{T}$. ♢

Pf. The Pythag thm, in form $\sin^2 = 1^2 - \cos^2$, gives

$$\begin{aligned}[4\cdot\text{Area}]^2 &\overset{\text{by (1)}}{=} [2\boldsymbol{ab}\cdot\sin(C)]^2\\ &\overset{\text{Pythag}}{=} [2\boldsymbol{ab}]^2 - [2\boldsymbol{ab}\cdot\cos(C)]^2\\ &\overset{\text{LoCos}}{=} [2\boldsymbol{ab}]^2 - [\boldsymbol{a}^2+\boldsymbol{b}^2-\boldsymbol{c}^2]^2.\end{aligned}$$

This last doesn't look symmetric in $\boldsymbol{a}$,$\boldsymbol{b}$,$\boldsymbol{c}$, but squaring, then adding, produces RhS($2i$), as desired. ♦

Inscribed radius and circum-radius. Let r and $\mathcal{O}$ denote the radius and center of the in-circle of $\mathbf{T} := \triangle ABC$. Evidently $\text{Area}(\triangle A\mathcal{O}B) = \frac{1}{2}r\boldsymbol{c}$. Adding this to the areas of $\triangle B\mathcal{O}C$ and $\triangle C\mathcal{O}A$ yields that $\text{Area}(\mathbf{T}) = \frac{1}{2}r\cdot[\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}]$. Equivalently

3.1: $$\text{InRadius}(\mathbf{T}) = \frac{2\cdot\text{Area}(\mathbf{T})}{\text{Perimeter}(\mathbf{T})}.$$

Now let R denote the radius of CircumCircle($\mathbf{T}$). Let 2γ denote the central angle $\measuredangle A\mathcal{O}B$ of the circle-arc **not** owning C. Dropping a perpendicular from $\mathcal{O}$ to chord $\overline{AB}$ we see that

$$\sin(\gamma) = \frac{\frac{1}{2}\boldsymbol{c}}{R} = \frac{\boldsymbol{c}}{2R}.$$

Thus

$$R = \frac{\boldsymbol{c}}{2\sin(\gamma)} = \frac{\boldsymbol{a}\cdot\boldsymbol{b}\cdot\boldsymbol{c}}{2\boldsymbol{ab}\cdot\sin(\gamma)}.$$

The Central-angle thm asserts that the inscribed $\measuredangle C$ equals γ. So (1) hands us

3.2: $$\text{CircumRadius}(\mathbf{T}) = \frac{\boldsymbol{a}\cdot\boldsymbol{b}\cdot\boldsymbol{c}}{4\cdot\text{Area}(\mathbf{T})}.$$

We get this curious corollary for the radii-ratio:

3.3: $$\frac{\text{CircumRadius}}{\text{InRadius}} = \frac{[\boldsymbol{a}\cdot\boldsymbol{b}\cdot\boldsymbol{c}]\,[\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}]}{8\cdot[\text{Area}^2]}.$$

Reciprocating, then using Heron's formula ($2i$), gives

3.4: $$\begin{aligned}\frac{2\cdot\text{InRadius}}{\text{CircumRad}} &= \frac{\big[-\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}\big]\big[\boldsymbol{a}-\boldsymbol{b}+\boldsymbol{c}\big]\big[\boldsymbol{a}+\boldsymbol{b}-\boldsymbol{c}\big]}{\big[\boldsymbol{a}\cdot\boldsymbol{b}\cdot\boldsymbol{c}\big]}\\ &= \Big[\frac{\boldsymbol{b}+\boldsymbol{c}}{\boldsymbol{a}}-1\Big]\Big[\frac{\boldsymbol{a}+\boldsymbol{c}}{\boldsymbol{b}}-1\Big]\Big[\frac{\boldsymbol{a}+\boldsymbol{b}}{\boldsymbol{c}}-1\Big].\end{aligned}$$

Boomerangs cannot tile a convex polygon

(Problem from David Gale.) A ***boomerang*** is a non-convex quadrilateral; call its $>\frac{\pi}{2}$ interior-angle **"fat"**. Conversely, a quadrilat(eral) with each angle $\leqslant\frac{\pi}{2}$ is a ***kite***. A dissection of a polygon **P** into *finitely many* quadrilats is a **"*quadritiling* of P"**. The tiles in a quadritiling *need not* be congruent to each other.

4.1: Boom-Kite Thm. *Each quadritiling of a convex polygon* **P** *must use a kite.* ◊

4.2: *Fails with "Quad" replaced by "Penta".* Let **P** be the square with vertices $(\pm 2, \pm 2)$. Cut **P** with a polygonal path going from/to

$$(2,2) \to (\text{-}1,1) \to (1,\text{-}1) \to (\text{-}2,\text{-}2)\,.$$

This cuts **P** (which is convex) into two non-convex pentagons (which are congruent to each other).

Exercise: Each polygon **Q**, convex or not, admits a (finite) tiling by non-convex *penta*gons. □

Nested convex curves

Attributed to Archimedes is the following theorem:

5: Theorem (Archimedes). *Suppose E and D are are compact convex sets in the plane, with $E \supset D$. Then the arclength* (of the boundary of) *E dominates the arclength of D.* ◊

Proof. Let a ***chord*** denote a line-segment having both its endpoints on ∂E and which is tangent to D. Cut off the "outside piece" of a chord from E to get the smaller convex body $E_1 \supset D$. Automatically, $\text{Len}(\partial E_1)$ is less-equal that of $\text{Len}(\partial E)$.

We can do a sequence of cuts to get a sequence of convex bodies $E \supset E_1 \supset E_2 \supset \ldots$, all of which are supersets of D. Moreover, we can arrange that the E_n "converge" to D –say, in the Hausdorff metric, or, even easier, in the sense that $\bigcap_{n=1}^{\infty} E_n = D$.

One can then show, since the objects are convex, that the arclength of ∂E_n is converging to $\text{Len}(\partial D)$. And $n \mapsto \text{Len}(\partial E_n)$ is a non-increasing function. ♦

Finding an equation of a circle, given a non-colinear triple of points.

[jk: The 4×4 Det is from `edgar@mps.ohio-state.edu`]

We have three points $(\boldsymbol{A},\boldsymbol{\alpha})$, $(\boldsymbol{B},\boldsymbol{\beta})$ and $(\boldsymbol{C},\boldsymbol{\gamma})$ in the plane and we let

$$\mathsf{E} \quad := \quad \begin{bmatrix} \boldsymbol{A} & \boldsymbol{\alpha} & 1 \\ \boldsymbol{B} & \boldsymbol{\beta} & 1 \\ \boldsymbol{C} & \boldsymbol{\gamma} & 1 \end{bmatrix}.$$

[More generally, the coordinates can come from an arbitrary commutative ring.]

6.1: Colinearity lemma. *The triple of points is colinear IFF* $\text{Det}(\mathsf{E}) = 0$. ◊

Pf of ($\Rightarrow$). The triple lies in a line, so the three rows of E, viewed as points in $\mathbb{R}^3$, lie in a lift of that line to the $z{=}1$ plane, hence lie in a line in $\mathbb{R}^3$. Consequently, Spn(Lifted points) is at-most 2-dimensional. ♦

Pf of ($\Leftarrow$). By hyp., the triple of E-rows [viewed as points in $\mathbb{R}^3$] lie in plane, P, through the origin. But they also lie in the $z{=}1$ plane; it misses the origin, so does *not equal* P. Hence the intersection of these two planes lies in a line in the $z{=}1$ plane; and this line projects to a line in $\mathbb{R}^2$. ♦

6.2: Circle-eqn lemma. *An equation* $\text{Fnc}(x,y) = 0$ *of the circle through the non-colinear triple of points can be given as a* 4×4 *determinant-eqn:*

$$6.3: \qquad \text{Det}\begin{bmatrix} x^2+y^2 & x & y & 1 \\ \boldsymbol{A}^2+\boldsymbol{\alpha}^2 & \boldsymbol{A} & \boldsymbol{\alpha} & 1 \\ \boldsymbol{B}^2+\boldsymbol{\beta}^2 & \boldsymbol{B} & \boldsymbol{\beta} & 1 \\ \boldsymbol{C}^2+\boldsymbol{\gamma}^2 & \boldsymbol{C} & \boldsymbol{\gamma} & 1 \end{bmatrix} \quad = \quad 0\,.$$ ◊

Proof. Expanding LhS(6.3) along the first row shows it to be a polynomial with a common coeff for x^2 and for y^2 of $\text{Det}(\mathsf{E})$; this latter is non-zero, courtesy (6.1). Hence (6.3) is the equation of *some* circle [which possibly is degenerate or empty].

Certainly $(x,y) := (\boldsymbol{A},\boldsymbol{\alpha})$ satisfies (6.3), since a matrix with two rows equal has Det=0. Ditto $(\boldsymbol{B},\boldsymbol{\beta})$ and $(\boldsymbol{C},\boldsymbol{\gamma})$ lie on the circle. Hence the circle is not degenerate, since a non-colinear triple lies on it. ♦

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