Generating Function examples

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Tools. Use GF for "generating function", and OGF/EGF for "Ordinary/Exponential GF". We derived in class, the following:

1.1: $$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n .$$

1.2: $$\frac{1}{[1-x]^2} = \sum_{k=0}^{\infty} [k+1] \cdot x^k .$$

More generally, for L a posint,

1.3: $$\frac{1}{[1-x]^L} = \sum_{k=0}^{\infty} \binom{k+L-1}{L-1} \cdot x^k .$$

1.4: $$\log\Bigl(\frac{1}{1-x}\Bigr) = \sum_{k=1}^{\infty} \tfrac{1}{k} \cdot x^k .$$

Consider $A(x) \overset{\text{OGF}}{\longleftrightarrow} \vec{a}$ and $B(x) \overset{\text{OGF}}{\longleftrightarrow} \vec{b}$. Recall that product $A(x)B(x)$ is the OGF of $\vec{c} := \vec{a} \circledast \vec{b}$, the ***convolution*** of $\vec{a}$ with $\vec{b}$, where

1.5: $$c_n := \sum_{j+k=n} [a_j \cdot b_k] ,$$

where (j,k) ranges over all ordered-pairs of natnums with $j+k = n$. As a special case, note $\vec{a} \circledast (1,1,\dots)$ is the partial-sum seq $\vec{c}$, where $c_n = \sum_{j=0}^{n} a_j$.

Counting Involutions in $\mathbb{S}_N$

Let t_N be the # of involuTions in the N^{th} symmetric group; permutations of token set $\Omega_N := \{\hat{1}, \hat{2}, \dots, \hat{N}\}$. Involutions are the perms composed only of 1-cycles and 2-cycles. Easily, $t_0 = 1$, $t_1 = 1$ and $t_2 = 2$.

Seq. $\vec{t}$ grows factorial-ishly because, just counting perms with the maximun number, $h := \lfloor \frac{N}{2} \rfloor$, of 2-cycles, shows that

2a: $$t_N \ \geq\ [N-1][N-3][N-5]\cdots[N-h] .$$

This, since $\hat{N}$ can be paired $[N-1]$ other tokens. Now the highest unpaired token has $[N-3]$ candidate tokens to be paired with; etc. Note that RhS(2a) dominates $[N-2][N-4][N-6]\cdots[\text{2 or 1}]$. Thus

2a′: $$t_N \ \geq\ \sqrt{[N-1]!}\ , \quad \textit{for all } N \geq 1.$$

2b: Lemma. *Involution-sequence $\vec{t}$ satisfies*

$$t_{n+2} = t_{n+1} + [n+1]t_n$$

for all natnums n. ◊

Proof. There are t_{n+1} involutions in $\mathbb{S}_{n+2}$ which fix token $\widehat{n+2}$, since the remaining tokens are permuted via an involution.

The other case is that $\widehat{n+2}$ is in 2-cycle. He can be paired with $n+1$ many other tokens, leaving n tokens to be involved. ♦

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% (involut 10)

 Use Low_k for Ceil(Sqrt(k!))
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n:	t_n	Low_nMO	Low_n	t_n/n!
0:	1	*	1	1
1:	1	1	1	1
2:	2	1	2	1
3:	4	2	3	2/3
4:	10	3	5	5/12
5:	26	5	11	13/60
6:	76	11	27	19/180
7:	232	27	71	29/630
8:	764	71	201	191/10080
9:	2620	201	603	131/18144
10:	9496	603	1905	1187/453600

OGF or EGF?. The rapid growth of $\vec{t}$, (2a), suggests using an EGF rather than an OGF. Define EGF

$$\mathbf{G} = \mathbf{G}(x) := \sum_{n=0}^{\infty} \frac{t_n}{n!} x^n . \quad \textit{Note that}$$

¥: $$x \cdot \mathbf{G}' = \sum_{n=0}^{\infty} n \cdot \frac{t_n}{n!} x^n .$$

2c: Thm. *The EGF of $\vec{t}$, the count-involutions sequence, is*

$$\mathbf{G}(x) = \exp\Bigl(\frac{x^2}{2} + x\Bigr) = \exp\Bigl(x \cdot \Bigl[\frac{x}{2} + 1\Bigr]\Bigr) . \quad ◊$$

Proof. Multiplying (2b) by x^{n+2} gives

$$t_{n+2}\, x^{n+2} \quad = \quad x \cdot t_{n+1}\, x^{n+1} \;+\; x^2 \cdot [n{+}1] t_n x^n \,.$$

Dividing by $[n{+}1]!$ produces

$$*: \quad [n{+}2]\cdot\frac{t_{n+2}}{[n{+}2]!} x^{n+2} \;=\; x\cdot\frac{t_{n+1}}{[n{+}1]!} x^{n+1} \;+\; x^2\cdot\frac{t_n}{n!} x^n \,.$$

Courtesy (¥), applying $\sum_{n=0}^{\infty}$ to LhS($*$) gives

$$[x\mathbf{G}'] - 0\cdot\tfrac{t_0}{0!} - 1\cdot\tfrac{t_1}{1!}x \;\stackrel{\text{note}}{=\!=}\; [x\mathbf{G}'] - x \,,$$

since $t_1 = 1$. And summing RhS($*$) hands us

$$x\cdot[\mathbf{G} - \tfrac{t_0}{0!}] \;+\; x^2\cdot\mathbf{G} \;\stackrel{\text{note}}{=\!=}\; x\mathbf{G} - x + x^2\cdot\mathbf{G} \,.$$

Equating these, then dividing by x, results in

2d: $$\mathbf{G}' \;-\; [x+1]\mathbf{G} \;=\; 0 \,.$$

This is a FOLDE (First-Order Linear DE), solved by antidifferentiating coefficient-fnc $-[x+1]$, then negating, producing $\frac{x^2}{2} + x$. Exponentiaing this gives $W(x) := \exp(\frac{x^2}{2} + x)$. All solns to the DE have form $\alpha\cdot W(x)$, for $\alpha \in \mathbb{C}$. We need *the* α such that

$$1 \;\stackrel{\text{note}}{=\!=}\; \tfrac{t_0}{0!} \;=\; \alpha \cdot W(0) \;\stackrel{\text{note}}{=\!=}\; \alpha \cdot 1 \,.$$

So $\alpha = 1$. ♦

Non-closed formula for t_n.

From (2c),

$$\mathbf{G}(x) \;=\; \sum_{k=0}^{\infty} \frac{1}{k!}\, x^k \Big[\frac{x}{2} + 1\Big]^k .$$

Courtesy the Binomial thm,

$$\Big[\frac{x}{2} + 1\Big]^k \;=\; \sum_{j=0}^{k} \binom{k}{j} \Big[\frac{x}{2}\Big]^j .$$

Hence $\mathbf{G}(x)$ equals

$$\sum_{k=0}^{\infty}\sum_{j=0}^{k} \frac{1}{k!}\frac{1}{2^j}\binom{k}{j} x^{j+k} \;=\; \sum_{k=0}^{\infty}\sum_{j=0}^{k} \frac{1}{[k-j]!\; j!\; 2^j}\, x^{j+k}.$$

Fix $n := j + k$. Then $k = n - j$ so $k - j = n - 2j$. Also, the largest value of j is $\lfloor n/2 \rfloor$, since $j \le k$. Thus, in the above sum, the coeff of x^n is

$$\sum_{j=0}^{\lfloor n/2 \rfloor} \frac{1}{[n-2j]!\; j!\; 2^j} \,.$$

We have established the following.

2e: Theorem. *For all natnums n,*

$$t_n = n!\cdot\sum_{j=0}^{\lfloor n/2 \rfloor} \frac{1}{[n{-}2j]!\; j!\; 2^j} \;\stackrel{\text{note}}{=\!=}\; \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{n{-}2j,\; j,\; j} \frac{j!}{2^j},$$

where $\binom{n}{n-2j,\, j,\, j}$ *denotes a multinomial coefficient.* ◊

Remark. The above summand

$$\binom{n}{n-2j,\; j,\; j} \cdot \frac{j!}{2^j}$$

has the combinatorial interpretation of counting the number of involutions with precisely j many 2-cycles.

Pocket-1 holds the $n{-}2j$ fixed-pts. The j many tokens in Pocket-2 will be paired with the j tokens in Pocket-3, and there are $j!$ many ways to do the pairing.

Finally, we've over-counted by a factor of 2^j since, for each pair, we can reverse which Pocket each is in. □

Matrix description of t_n.

With matrix

$$\mathsf{M}_n \;:=\; \begin{bmatrix} 0 & 1 \\ n & 1 \end{bmatrix} \quad \textit{and column-vector} \quad \mathbf{v}_n \;:=\; \begin{bmatrix} t_n \\ t_{n+1} \end{bmatrix},$$

we can restate recurrence (2b) as

2f: $$\mathbf{v}_{n+1} \;=\; \mathsf{M}_{n+1} \cdot \mathbf{v}_n \,.$$

Hence $\mathbf{v}_n = \mathsf{M}_n\mathsf{M}_{n-1}\cdots\mathsf{M}_2\mathsf{M}_1\mathbf{v}_0$. And $\mathbf{v}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, so

$$\begin{bmatrix} t_n \\ t_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ n & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ n{-}1 & 1 \end{bmatrix}\cdots\begin{bmatrix} 0 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Observations. Since $\mathrm{Det}(\mathsf{M}_n) = -n$, we have

$$\mathrm{Det}(\mathsf{M}_n\mathsf{M}_{n-1}\cdots\mathsf{M}_2\mathsf{M}_1) \quad = \quad [-1]^n \cdot n! \,.$$

The char-poly of M_n is

$$\wp(z) \;=\; z^2 - z - n \;=\; [z - \lambda_n^+] \cdot [z - \lambda_n^-] \,,$$

where the eigenvalues of M_n are

$$\lambda_n^{\pm} \;:=\; \tfrac{1}{2}\big[1 \pm \sqrt{1+4n}\,\big]\,, \quad \textit{with}$$
$$\lambda_n^{+} + \lambda_n^{-} \;=\; 1 \qquad \textit{and} \qquad \lambda_n^{+}\cdot\lambda_n^{-} \;=\; -n\,.$$

Corresponding eigenvectors are

$$\mathbf{e}_n^{+} \;:=\; \begin{bmatrix}\lambda_n^{-}/\,n \\ -1\end{bmatrix} \quad \textit{and} \quad \mathbf{e}_n^{-} \;:=\; \begin{bmatrix}\lambda_n^{+}/\,n \\ -1\end{bmatrix}.$$

Unfinished: as of 12May2024

Remark.

Do eigenvalues bound expand/shrink rate

Consider 2×2 matrix M. Let $\|\mathsf{M}\|_{\text{op}}$ denote the *maximum*, taken over all unit-vectors $\mathbf{v}$ (i.e $\|\mathbf{v}\| = 1$), of the ratio

$$*:\qquad \frac{\|\mathsf{M}\mathbf{v}\|}{\|\mathbf{v}\|}\,.$$

This $\|\mathsf{M}\|_{\text{op}}$ is called the "***operator norm*** of M".

For number $\mathcal{S}>0$ to be determined later, define

$$\mathsf{E} \;:=\; \begin{bmatrix}\mathcal{S}^2+\mathcal{S} & \mathcal{S}^3-\mathcal{S}^2 \\ \mathcal{S}-1 & \mathcal{S}^2+\mathcal{S}\end{bmatrix}.$$

Verify that

$$3:\qquad \mathbf{a} \;:=\; \begin{bmatrix}\mathcal{S}\\1\end{bmatrix} \quad \textit{and} \quad \mathbf{b} \;:=\; \begin{bmatrix}-\mathcal{S}\\1\end{bmatrix}$$

are E-eigenvectors, with respective eigenvalues

$$\alpha \;:=\; 2\mathcal{S}^2 \quad \textit{and} \quad \beta \;:=\; 2\mathcal{S}\,.$$

But for unit-vector $\mathbf{u} := \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$, product

$$\mathsf{E}\mathbf{u} \;=\; \begin{bmatrix}\mathcal{S}^3-\mathcal{S}^2 \\ \mathcal{S}^2+\mathcal{S}\end{bmatrix}.$$

So ratio $\frac{\|\mathsf{E}\mathbf{u}\|}{\|\mathbf{u}\|} \approx \mathcal{S}^3$. For large $\mathcal{S}$, then, this ratio is much larger than α, the largest eigenvalue.

Similarly, for $\mathbf{w} := \begin{bmatrix}1-\mathcal{S} \\ [\mathcal{S}+1]/\mathcal{S}\end{bmatrix}$, note

$$\mathsf{E}\mathbf{w} \;=\; \begin{bmatrix}0\\4\mathcal{S}\end{bmatrix}.$$

For large $\mathcal{S}$, note $\|\mathbf{w}\| \approx \mathcal{S}$, so $\frac{\|\mathsf{E}\mathbf{w}\|}{\|\mathbf{w}\|} \approx 4$. □

Warm-up. Consider sequence $\vec{\boldsymbol{a}} := (a_0, a_1, a_2, \dots)$ where

$$\dagger:\qquad a_0 \;:=\; 0 \quad \textit{and} \quad a_{n+1} \;:=\; 1 + 3a_n\,.$$

We'll get a formula for a_n, by manipulating the *entire sequence* $\vec{\boldsymbol{a}}$. The method is to define the corresponding ***(ordinary) generating function*** [abbrev. ***OGF***]

$$A(x) \;:=\; \sum_{n=0}^{\infty} a_n\cdot x^n\,.$$

The RhS, here, converges for all $|x| < \frac{1}{3}$. However, it turns out that, for most problems, we can view the RHS as a *formal power series*, and never worry about convergence.

Let's manipulate OGF $A(x)$ so as to get a recurrence formula for it. Note first that

$$*:\quad \sum_{n=0}^{\infty} a_{n+1}\cdot x^n \;=\; \Big[\sum_{n=0}^{\infty} 1\cdot x^n\Big] + 3\Big[\sum_{n=0}^{\infty} a_n x^n\Big] \;=\; \tfrac{1}{1-x} + 3A(x)\,.$$

OTOHand, product $x\cdot\text{LhS}(*)$ equals

$$x\cdot\sum_{n=0}^{\infty} a_{n+1}\cdot x^n \;=\; \sum_{k=1}^{\infty} a_k\cdot x^k \;=\; A(x)\,,$$

since $a_0 = 0$. Thus, multiplying $(*)$ by x, gives

$$A \;=\; x\cdot\big[\tfrac{1}{1-x} + 3A\big] \;=\; \tfrac{x}{1-x} + 3xA\,.$$

Hence $A\cdot[1-3x] = \frac{x}{1-x}$, so $A = \frac{x}{[1-3x][1-x]}$. The *partial-fraction* decomposition now yields

$$A \;=\; \frac{1/2}{1-3x} - \frac{1/2}{1-x} \;=\; \frac{1}{2}\Big[\Big[\sum_{n=0}^{\infty}[3x]^n\Big] - \Big[\sum_{n=0}^{\infty} x^n\Big]\Big].$$

Combining terms gives the power series expansion,

$$A(x) \;=\; \sum_{n=0}^{\infty} \tfrac{1}{2}[3^n - 1]\cdot x^n\,.$$

And you can easily check that the formula

$$a_n \;:=\; \tfrac{1}{2}[3^n - 1]\,, \quad \text{for } n = 0,1,2,\dots,$$

indeed satisfies the original $(\dagger)$ recurrence.

Unless mentioned otherwise, the following problems are from Bona's text.

5.1: #**48**[P.]*207.* *Let g_n be the number of (combinatorial) simple graphs on $[1..n]$ in which each vertex has degree 2. With $G(x) \overset{\text{EGF}}{\longleftrightarrow} \vec{\boldsymbol{g}}$, prove that*

$$G(x) \;=\; \frac{1}{\sqrt{1-x}} \cdot \mathrm{e}^{-\frac{x}{2}-\frac{x^2}{4}} . \qquad \Diamond$$

Soln. On a k-set, let a_k be the number of cyclic simple graphs using all k of the vertices. So $a_0 = a_1 = a_2 = 0$ (The $a_0 = 0$ needs comment). For $k \geq 3$, there are $[k-1]!$ circular permutations ♦

6.1: #**43**[P.]*176* (People). *Let q_n be the number of ways of partitioning n people into groups labeled "E", "D" and "A", and asking each group to form a line, where group E has an even number of folk, group D oddly many, and group A has an arbitrary number of people. Get a closed-formula for q_n.* ◊

Ans. We'll give an answer first, then show three derivations. The N^{th} ***triangular number*** is

$$\tau_N \;\;:=\;\; \sum_{k=1}^{N} k \;\;=\;\; \tfrac{1}{2} \cdot N[N+1] .$$

Define a sequence $\vec{\boldsymbol{s}} = (s_0, s_1, \dots)$ by

$$\text{6.2:} \quad \vec{\boldsymbol{s}} \;=\; (\tau_0, \tau_1, \tau_1, \tau_2, \tau_2, \tau_3, \tau_3, \tau_4, \tau_4, \dots) \;=\; (0, 1, 1, 3, 3, 6, 6, 10, 10, \dots) .$$

Then, for each natnum n,

$$\text{6.3:} \quad q_n \;=\; n! \cdot s_n . \qquad \square$$

1st Soln. For the three groups, the corresponding EGFs are

$$A(x) \;:=\; \sum_{n=0}^{\infty} \frac{n!}{n!} \cdot x^n \;\overset{\text{note}}{=\!=}\; \frac{1}{1-x} ;$$

$$E(x) \;:=\; \sum_{j=0}^{\infty} 1 \cdot x^{2j} \;\overset{\text{note}}{=\!=}\; \frac{1}{1-[x^2]} ;$$

$$D(x) \;:=\; \sum_{k=0}^{\infty} 1 \cdot x^{2k+1} \;\overset{\text{note}}{=\!=}\; \frac{x}{1-[x^2]} .$$

Hence the EGF for $\vec{\boldsymbol{q}}$ is $Q := A \cdot E \cdot D$. I.e

$$\text{6.4:} \qquad Q(x) \;=\; \frac{x}{[1-x^2]^2} \cdot \frac{1}{1-x} .$$

This RhS(6.4) is the OGF of some sequence $\vec{\boldsymbol{s}}$. Tool (1.2) says $\frac{1}{[1-y]^2} \overset{\text{OGF}}{\longleftrightarrow} (1, 2, 3, \dots)$. Hence

$$(0, 1, 0, 2, 0, 3, 0, 4, 0, \dots) \overset{\text{OGF}}{\longleftrightarrow} \frac{x}{[1-x^2]^2} .$$

Convolving this sequence with

$$(1, 1, 1, \dots) \overset{\text{OGF}}{\longleftrightarrow} \frac{1}{1-x}$$

forms the partial sums of $(0, 1, 0, 2, \dots)$. Hence (6.2). Finally, $Q(x)$ is the EGF of $\vec{\boldsymbol{q}}$, whence (6.3). ♦

2nd Soln. This problem is a fancier version of **#14**$^{\text{P.}176}$, the bookshelf. Make a line of books by choosing one of the $n!$ orderings. Let s_n be the number of ways of:

> *Taking some even number from the left of the line, and putting them on shelf "E", then taking some oddly many from the left of what remains, and putting that on shelf "D".*

Hence (6.3), and we just need to compute $\vec{s}$.

In the plane, consider the triangle of lattice-points,

$$\Omega_n \;:=\; \Big\{(x,y) \in \mathbb{N}{\times}\mathbb{N} \;\Big|\; x{+}y \le n\Big\}.$$

Interpret a point $(\varepsilon, \delta) \in \Omega_n$ as putting ε many books on "E", then δ many books on "D", finally $n - [\varepsilon{+}\delta]$ many books on the "Arbitrary" shelf. If ε is even and δ odd, then this is a valid placement. Thus

6.5: *Our s_n is the number of $(\text{Even}, \text{Odd})$ points in the Ω_n lattice-triangle.*

Approximately half have even x-coordinate, about half have odd y, and these two events are more-or-less independent. Conclusion: $\boxed{s_n \text{ is approx } \frac{n^2}{2}/4}$. Also, a valid (ε, δ) has $\varepsilon{+}\delta$ odd. E.g, the line of lattice-pts (x, y) with $x{+}y = 6$ has no valid points. Hence $s_6 = s_5$. More generally, $\boxed{n \text{ odd} \Rightarrow s_n = s_{n+1}}$. Of course, our (6.2) implies both of the circled stmts.

Counting valid pts. Fix an odd $n = 2k - 1$, and consider those $(x, y) \in \mathbb{N}{\times}\mathbb{N}$ with $x{+}y = n$. Certainly x is even IFF y is odd, so precisely k of those points are valid. It follows that $\vec{s}$ comprises the partial sums of sequence $(0, 1, 0, 2, \ldots)$, as n takes on values $0, 1, 2, 3, \ldots$. So we have again derived (6.2). ♦

3rd Soln. Back to GFs!

The partial-fraction decom of RhS(6.4) is

$$\frac{-1}{16} \cdot \left[\frac{1}{1-x} + \frac{1}{1+x} + \frac{4}{[1-x]^3} + \frac{2}{[1+x]^2}\right].$$

This looks arduous to do directly, so let's finesse things. The "$1 - x$" and "$1 + x$" will cause even/odd index terms to behave differently, so let's separate them. Write $\sum_{n=0}^{\infty} s_n x^n := Q(x) = \varepsilon(x) + \delta(x)$, where

$$\varepsilon(x) \;:=\; \sum_{n \text{ even}} s_n x^n \quad \text{and} \quad \delta(x) \;:=\; \sum_{n \text{ odd}} s_n x^n.$$

Now $2\varepsilon(x) = Q(x) + Q(-x)$, which equals

$$\dagger\!: \quad \frac{1}{[1-x^2]^2} \cdot \left[\frac{x}{1-x} + \frac{-x}{1+x}\right] \;=\; \frac{2x^2}{[1-x^2]^3}.$$

So $\varepsilon(x) = f(x^2)$, where $f(y) := \frac{y}{[1-y]^3}$. By (1.3),

$$f(y) \;=\; y \cdot \sum_{i=0}^{\infty} \binom{i+2}{2} y^i \;\overset{\text{note}}{=\!=}\; \sum_{k=1}^{\infty} \tau_k \cdot y^k,$$

by setting $k = i{+}1$. And $\tau_0 = 0$, so

$$\varepsilon(x) \;=\; \textstyle\sum_{k=0}^{\infty} \tau_k \cdot x^{2k};$$

this justifies the even-indexed terms in (6.2).

For the odd-index terms, $2\delta(x) = Q(x) - Q(-x)$ which equals

$$\ddagger\!: \quad \frac{1}{[1-x^2]^2} \cdot \left[\frac{x}{1-x} - \frac{-x}{1+x}\right] \;=\; \frac{2x}{[1-x^2]^3}.$$

Comparing with (†), then, $x \cdot \delta(x) = \varepsilon(x)$. I.e, when n is an *even* index, then $s_{n-1} = s_n$. Hence (6.2). ♦

7.1: #44 P 176. *From n people, select a committee of oddly many. From the committee, select a council of evenly many [allowing the value zero]. Get a closed-formula for r_n, the number of ways of doing this.* ♢

Prelim. View this as splitting the people into

- Group E, the council with evenly many.
- Group D with oddly many, where the committee is $\mathrm{E} \sqcup \mathrm{D}$.
- Group A, with arbitarily many; those that remain.

While superficially similar to #43 P 176, the lack of ordering makes a difference. A seat-of-the-pants growth estimate is

7.2: $$3^n \geq r_n \geq 2^{n-1} - 1 .$$

The first follows by removing the even/odd restrictions, so each elt of $[1 .. n]$ admits 3 colors.

The lower bnd holds for $n=0$; what about $n \geq 1$? Well, $[1 .. n]$ has 2^{n-1} even-cardinality subsets. Hence there are at least $[2^{n-1} - 1]$ even subsets that are not all of $[1 .. n]$; and so we can pick one element of the complement to make a singleton D. □

Soln. With $R(x) \overset{\text{EGF}}{\longleftrightarrow} \vec{r}$, we wish to define EGFs so that $R(x) = E(x)D(x)A(x)$. So $E(x) \overset{\text{EGF}}{\longleftrightarrow} \vec{e}$, where e_k is the number of ways making an even-sized council using all k people. I.e, $\vec{e} = (1, 0, 1, 0, \ldots)$. Thus

$$E(x) = \sum_{k \text{ even}} \frac{x^k}{k!} = \tfrac{1}{2}[\mathrm{e}^x + \mathrm{e}^{-x}] . \quad \textit{Similarly,}$$

$$D(x) = \sum_{k \text{ odd}} \frac{x^k}{k!} = \tfrac{1}{2}[\mathrm{e}^x - \mathrm{e}^{-x}] .$$

So $E(x)D(x) = \frac{1}{4}[\mathrm{e}^{2x} - \mathrm{e}^{-2x}]$. Since $A(x) = \mathrm{e}^x$, our $R(x)$ is $\frac{1}{4}[\mathrm{e}^{3x} - \mathrm{e}^{-x}]$. Thus for each natnum n,

7.3: $$r_n = \tfrac{1}{4} \cdot \big[3^n - [-1]^n\big] .$$

BTWay, $3^n - [-1]^n \equiv_4 [-1]^n - [-1]^n = 0$, as it must. ♦

7.4: *Rem.* Curiously, (7.3) is a sum of exponentials, thus satifies a 2-term linear recurrence. The two bases, 3 and -1, are roots of the polynomial

$$f(x) \coloneqq [x-3][x+1] \overset{\text{note}}{=\!=} x^2 - 2x - 3 .$$

Each base $b \in \{3, -1\}$ satisfies $f(b)=0$, i.e $b^2 = 2b + 3$, and thus $b^{n+2} = 2b^{n+1} + 3b^n$, for each n. Consequently, $\vec{r}$ satisfies recurrence

7.5: $$r_{n+2} = 2r_{n+1} + 3r_n .$$

Exer: *Find a bijective proof of (7.5).* □

8.1: #**38**P.176. *Let g_n be the number of ways of selecting a permutation of $[1..n]$, then marking a particular cycle in the permutation. Obtain a formula for g_n.* ♢

Rem. Bona's Ex**8.25**P.171 asked how many ways to seat n people around circular tables. The answer was $n!$ [reminding us of the Canonical cycle notation], derived from Thm**8.24**P.171. Equivalent, is to use Thm**8.28**P.172 but with the outer seq trivial, the constant-1 sequence, because given j cycles, there is only one way to take all of them.

In the current problem, our outer sequence is $(0, 1, 2, \ldots)$, since there are j ways to pick one cycle from j. □

Soln. With $c_0 := 0$, henceforth let c_k be the number of cyclic-permutations of a k-set; so $c_k = [k-1]!$. The EGF of $\vec{c}$ is thus

$$C(x) := \sum_{k=1}^{\infty} \frac{[k-1]!}{k!} x^k = \sum_{k=1}^{\infty} \frac{1}{k} \cdot x^k = \log\left(\frac{1}{1-x}\right),$$

by (1.4). The EGF of picking one object from j is

$$B(x) := \sum_{j=1}^{\infty} \frac{j}{j!} \cdot x^j \stackrel{\text{note}}{=\!=} x \cdot \mathrm{e}^x .$$

Thm**8.28**P.172 now says $\vec{g}$ has EGF

$$B(C(x)) = \log\left(\frac{1}{1-x}\right) \cdot \frac{1}{1-x}$$
$$\stackrel{\text{OGF}}{\longleftrightarrow} \left(0, 1, \tfrac{1}{2}, \tfrac{1}{3}, \ldots\right) \circledast (1, 1, \ldots) .$$

The partial-sum seq of $(0, 1, \frac{1}{2}, \frac{1}{3}, \ldots)$ is the harmonic number seq, $(\mathsf{H}_0, \mathsf{H}_1, \mathsf{H}_2, \ldots)$, where

8.2: $$\mathsf{H}_n := \sum_{\ell=1}^{n} 1/\ell .$$

Consequently,

8.3: $$g_n = n! \cdot \mathsf{H}_n .$$ ♦

Evidence? Let's compute g_3 by hand.

i: *From three 1-cycles, pick one;* **3** *choices.*

ii: *There are* $\binom{3}{1} = 3$ *ways to split into a 2-cycle and a 1-cycle. For each, we have 2 ways to select a cycle, giving* $3 \cdot 2 = \mathbf{6}$ *choices.*

iii: *We can just have a single 3-cycle, and we must choose it. This gives* **1** *choice.*

Thus $g_3 = \mathbf{3}+\mathbf{6}+\mathbf{1} = 10$. We now use (8.3) to reassure ourselves, by computing $3! \cdot [1 + \frac{1}{2} + \frac{1}{3}] = 6 + 3 + 2$, which equals... *Eleven?!* *Oh no! –Mathematics is inconsistent! -oh, woe is me, oh...*

Wait a darn minute!; n objects can have many different cyclic permutations. For cycles of length 1 or 2, the cyclic ordering is unique. But in case (iii), above, there are 2 cyclic permutations of three objects. So my computation *should* have been: $g_3 = \mathbf{3} + \mathbf{6} + \mathbf{2}$, which –whew!– indeed equals eleven. □

9.1: #23$^{P.}$*176. Find a formula for a_n, where $a_0 := 1$ and $a_{n+1} = 3a_n + 2^n$.* ♢

Soln. Set $A(x) \overset{\text{OGF}}{\longleftrightarrow} \vec{\boldsymbol{a}}$. Multiply the recurrence by x^{n+1} and sum, to get that $A(x) - 1 = A(x) - a_0$ equals $3xA(x) + x \cdot \sum_{n=0}^{\infty} [2x]^n$. Hence

$$[1 - 3x] \cdot A(x) = 1 + \frac{x}{1-2x} \overset{\text{note}}{=\!=} \frac{1-x}{1-2x}.$$

So

$$A(x) = \frac{1-x}{[1-3x][1-2x]} = \frac{2}{[1-3x]} - \frac{1}{[1-2x]} = \sum_{n=0}^{\infty} \big[[2 \cdot 3^n] - 2^n\big] x^n.$$

Hence $\boxed{a_n = [2 \cdot 3^n] - 2^n}$, for each natnum n. ♦

10.1: #14$^{P.}$*176* (Books). *Let t_n be the number of ways of placing n books, some on shelf A, some on shelf B, with at least one book one each shelf. Obtain a "closed formula" for t_n.* ♢

Soln. Since the books can be split *arbitrarily* between the two shelves, we'd like to take a product of EGFs.

Let $A(x)$ be the EGF of seq $\big(0, 1!, 2!, \ldots\big)$, arranging books [at least one] on shelf A; so

$$A(x) = \sum_{j=1}^{\infty} \frac{j!}{j!} x^j \overset{\text{note}}{=\!=} \frac{x}{1-x}.$$

Similarly, the EGF of arranging books on shelf B is $B(x) = \frac{x}{1-x}$. With $T(x)$ the EGF of $\vec{\boldsymbol{t}}$, then,

$$T(x) = A(x)B(x) = x^2 \cdot \frac{1}{[1-x]^2} = x^2 \cdot \sum_{\ell=0}^{\infty} \binom{\ell+1}{1} x^\ell = \sum_{n=2}^{\infty} [n-1] x^n.$$

For $n \geq 2$, then, $t_n / n! = n-1$. Consequently,

10.2: $$t_0 = t_1 = 0, \text{ and then } t_n = n! \cdot [n-1].$$ ♦

10.3: *Remark.* Now we have (10.2), we can see a direct argument. Pick one of $n!$ orderings of all the books, then put a separator at any one of the $n-1$ junctures between adjacent books. Those on the separator's left, go on shelf A. □

The partition function. Let $\wp(n)$ be the number of partitions of integer n. E.g, the five ptns of 4 are 1+1+1+1, 1+1+2, 1+3, 2+2, 4; so $\wp(4) = 5$. For n negative, $\wp(n) = 0$. And $\wp(0) = 1$. For the partition 1+1+3+4 of nine, the summands $1, 1, 3, 4$ are called the ***parts*** of the partition.

Recall from class [or pages 98–101 of Bona] the ***Ferrers diagram*** of a ptn, and the ***conjugate*** (I also call it the ***transpose***) of a partition.

Interpret picking the k^{th}-term from sum

$$1 + x^3 + [x^3]^2 + [x^3]^3 + \ldots + [x^3]^k + \ldots \overset{\text{note}}{=\!=} \frac{1}{1 - x^3},$$

as having k copies of the part 3. Consequently,

11a: $$P(x) := \prod_{j=1}^{\infty} \frac{1}{1 - x^j}$$

is the OGF of $[n \mapsto \wp(n)]$. More generally, fix a subset $S \subset \mathbb{Z}_+$ and let $\wp_S(n)$ be the number of ptns of n using only parts from S. Then

11b: $$P_S(x) := \prod_{j \in S} \frac{1}{1 - x^j} \overset{\text{OGF}}{\longleftrightarrow} [n \mapsto \wp_S(n)].$$

12.1: #10 $^{P.}$174 (LargestPart=4). *Let b_n be the number of n-partitions whose largest part is 4. Compute the* OGF, *$B(x)$, of $\vec{\boldsymbol{b}}$.* ♢

Soln. Picking only size-4 parts, and at least one such, has OGF $[x^4 + x^8 + x^{12} + \ldots]$, which is $\frac{x^4}{1-x^4}$. Hence

$$B(x) \;=\; x^4 \cdot \prod_{k=1}^{4} \frac{1}{1 - x^k}.$$ ♦

13.1: #11 $^{P.}$105 (Equal-largest). *Let e_n be the number of n-ptns whose two largest parts are equal.* [So $e_1 = 0$, $e_2 = 1$, $e_3 = 1$, $e_4 = 2$.] *Prove that*

∗: $$\forall n \in \mathbb{Z}_+ : \quad e_n \;=\; \wp(n) - \wp(n{-}1).$$ ♢

Soln. [Rather than the injection argument of Thm 5.20 $^{P.}$101, let's use GFs.] Since $\wp(-1) = 0$, the OGF of $[n \mapsto \wp(n{-}1)]$ is

$$C(x) := \sum_{n=1}^{\infty} \wp(n{-}1) \cdot x^n \overset{\text{note}}{=\!=} x \cdot P(x).$$

Courtesy (11a), then,

$$P(x) - C(x) \;=\; [1 - x] \cdot P(x) \;=\; \prod_{j=2}^{\infty} \frac{1}{1 - x^j}.$$

By (11b), this is $P_S(x)$ where $S := [2 .. \infty)$. And the transpose of an S-ptn is a ptn that either has *no* parts [i.e, $n = 0$] or its largest two parts are equal. Defining $e_0 := 1$, then, we've shown that $P(x) - C(x)$ equals the OGF of $\vec{\boldsymbol{e}}$. Hence (∗). ♦

14.1: #**40** P 176 (Derangements). *Let d_n be the number of derangements of $[1..n]$. Compute $D(x)$, the EGF of $\vec{d}$.* ♢

Defn. A ***derangement*** of a set, is a fixed-point–free permutation of the set. So the above $\vec{d}$ has $d_0 = 1$, $d_1 = 0$, $d_2 = 1$ and $d_3 = 2$. □

Soln. A permutation is a derangement IFF each cycle has length$\geq$2. Set $c_0 = c_1 = 0$ and, for $k \geq 2$, let c_k be the number of cyclic-permutations of a k-set; so $c_k = [k-1]!$. The EGF of $\vec{c}$ is thus

$$C(x) := \sum_{k=2}^{\infty} \frac{[k-1]!}{k!} x^k = \Big[\sum_{k=1}^{\infty} \frac{1}{k} \cdot x^k\Big] - x .$$

By (1.4), then, $C(x) = \log\Big(\frac{1}{1-x}\Big) - x$. Our Exponential Thm [Thm 8.24 P 171] now implies that

$$\text{14.2:} \qquad D(x) = \mathrm{e}^{C(x)} = \frac{1}{1-x} \cdot \mathrm{e}^{-x} .$$ ♦

14.3: #**41** P 176 (More derangements). *For $n \in \mathbb{Z}_+$, prove that*

$$\text{14.4:} \qquad d_n - [n \cdot d_{n-1}] = [-1]^n .$$ ♢

Soln. Set $b_k := k \cdot d_{k-1}$; so $b_0 := 0$. Set $a_n := [-1]^n$. Let $B(x)$ and $A(x)$ be the EGFs of $\vec{b}$ and $\vec{a}$. Thus (14.4) will follow from

$$\text{14.5:} \qquad D(x) - B(x) \stackrel{?}{=} A(x) .$$

Computing. So $A(x) = \sum_{n=0}^{\infty} \frac{[-1]^n}{n!} x^n = \mathrm{e}^{-x}$. And

$$B(x) = \sum_{k=1}^{\infty} \frac{k \cdot d_{k-1}}{k!} x^k \stackrel{\text{note}}{=\!=} x \cdot \sum_{n=0}^{\infty} \frac{d_n}{n!} x^n = x \cdot D(x) .$$

Thus $D(x) - B(x)$ equals $[1-x]D(x)$ which, courtesy (14.2), equals e^{-x}. ♦

15.1: Ex **8.26** P 172. *Let t_n be the number of partitions of an n-set into atoms, each of cardinality 3. Get a closed formula for t_n.* ♢

Rem. When $n \nmid 3$, then $t_n = 0$. With $T(x) \stackrel{\text{EGF}}{\longleftrightarrow} \vec{t}$, then,

$$*: \qquad T(x) = \sum_{k=0}^{\infty} \frac{c_k}{[3k]!} \cdot x^{3k} ,$$

where $c_k := t_{3k}$. □

1st Soln. The neat soln in Bona's text: Let b_n be the number of ptns of an n-set, using a single atom of cardinality 3. So $b_3 = 1$, and every other b_n is zero. Thus $\vec{b} \stackrel{\text{EGF}}{\longleftrightarrow} x^3/3! =: B(x)$. So our Thm **8.24** P 171 says $T(x)$ equals

$$\mathrm{e}^{B(x)} \stackrel{\text{note}}{=\!=} \sum_{k=0}^{\infty} \frac{1}{k!} \cdot B(x)^k = \sum_{k=0}^{\infty} \frac{1}{k!} \cdot \frac{x^{3k}}{[3!]^k} .$$

Equating terms with $(*)$ yields

$$\text{15.2:} \qquad c_k = \frac{[3k]!}{k! \cdot [3!]^k} .$$ ♦

Amusingly, it is not even evident that the RhS is an integer...

"Bare hands" Soln. For a $k>0$, consider a valid ptn of $[1..n]$, where $n := 3k$. For the other two members of the atom owning n, there are $\binom{n-1}{2}$ choices. Consequently,

$$c_k = \binom{n-1}{2} \cdot c_{k-1} .$$

Since $c_0 = 1$, iterating gives a product of k terms,

$$c_k = \binom{n-1}{2} \cdot \binom{n-4}{2} \cdot \binom{n-7}{2} \cdots \binom{5}{2} \cdot \binom{2}{2} .$$

So $[2!]^k \cdot c_k$ equals

$$\mathit{1} \cdot [n-1] \cdot [n-2] \cdot \mathit{1} \cdot [n-4] \cdot [n-5] \cdot \mathit{1} \cdot [n-7] \cdot [n-8] \cdots \mathit{1} \cdot 5 \cdot 4 \cdot \mathit{1} \cdot 2 \cdot 1 ,$$

where I have put an italic-1 in front of each group. Replacing these *1*s successively by $n, n-3, n-6, \ldots, 6, 3$ multiplies this product by $3^k \cdot k!$, and thus:

$$3^k \cdot k! \cdot [2!]^k \cdot c_k = n! \stackrel{\text{note}}{=\!=} [3k]! .$$

Since 3^k times $[2!]^k$ is $[3!]^k$, we can rewrite this as

$$k! \cdot [3!]^k \cdot c_k \quad = \quad [3k]!\,.$$

Solving for c_k now gives (15.2). But, *Oy!*, this was so much more work.... ♦