Staple!

Team: ____

Advanced Calc
MAA4102 & MAA5104

Home-γ

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Definitions. Given a set X, a function f defined on X, and a sequence $\mathbf{z} := (z_n)_{n=1}^{\infty} \subset X$, let "$f(\mathbf{z})$" mean the sequence whose n^{th} term is $f(z_n)$.

For an algebraic number α, recall that its ***degree***, written $\text{Deg}(\alpha)$, is the degree of the smallest-degree non-zip intpoly f for which $f(\alpha) = 0$. (So the degree of a rational number is 1.) If α is transcendental, then let $\text{Deg}(\alpha) := \infty$.

γ1: Let d be the usual Euclidean metric on the plane, $X := \mathbb{R}\times\mathbb{R}$. Let U and C be the open and closed unit-balls, in X, centered at the origin. Suppose that $f\colon U\to\mathbb{R}$ is a *uniformly* continuous function.

___ ___ a Prove: *For each point $p \in \partial U$, there is a sequence $\mathbf{x} \subset U$ with $\lim(\mathbf{x}) = p$.* [*Hint:* This is trivial. Give an explicit sequence in terms of p.]

___ ___ b Prove: *If $\mathbf{x} \subset U$ and $\lim(\mathbf{x}) = p$, then $\lim\big(f(\mathbf{x})\big)$ exists in $\mathbb{R}$.* [*Hint:* What does a uniformly continuous function do to a Cauchy sequence? Prove it!]

___ ___ c For each point $p \in \partial U$ and each two seqs $\mathbf{x}, \mathbf{y} \subset U$ which converge to p, show that sequences $f(\mathbf{x})$ and $f(\mathbf{y})$ necessarily have the same limit; call it $\widehat{p}$.

Now define a new function $g\colon C\to\mathbb{R}$ by: *For $y \in U$, let $g(y) := f(y)$. For $p \in C \setminus U$, let $g(p)$ be the value $\widehat{p}$.*

___ ___ d Prove that g is cts at each point p in $\partial U \overset{\text{note}}{=\!=} C \setminus U$ [*Hint:* If you choose to consider sequences $\mathbf{x} \subset C$ converging to p, note that such a sequence $\mathbf{x}$ might not lie entirely within U.]

Extra Credit: Prove something stronger about g.

γ2: Say that a real number α is ***rapidly approximable*** **(*by rationals*)** if there exist pairs of integers p and q, with q as large as desired, so that

1: $$0 < |\alpha - \tfrac{p}{q}| < 17/q^3 .$$

i Use the Mean Value Theorem to prove that if α is rapidly approximable, then $\text{Deg}(\alpha) > 2$.

[*Hint:* FTSOC suppose that f is a non-zip intpoly, with $\text{Deg}(f) \leqslant 2$, for which $f(\alpha) = 0$. Apply the MVT to f to write

2: $$f(\tfrac{p}{q}) - f(\alpha) = \left[\tfrac{p}{q} - \alpha\right] \cdot f'(c)$$

for some point c between $\frac{p}{q}$ and α. (Of course, c *depends on* the rational number $\frac{p}{q}$.) Now take absolute-values on each side of **2:**, and then multiply each side by q^2. What kind of number is the new LHS? What can you dominate the RHS by?] [*Random thought:* Since f is non-zip, this f has only finitely many roots.]

[*Hint:* When q is large then $\frac{p}{q}$ is close to α; so c is close to α. Thus $f'(c)$ is near $f'(\alpha)$. (*Why?*) In particular, as $\frac{p}{q} \to \alpha$, the numbers $f'(c)$ do *not* blow up to ∞.]

ii Generalize part (i) by replacing "3" by something.

[*A bit of Extra Credit:* Use your generalization to construct, provably, a particular transcendental number τ. If you wish, you may prove that

$$\tau := \sum_{n=1}^{\infty} \frac{1}{b_n}, \quad \text{where } b_n := 2^{n!},$$

is transcendental. Note:The above exponent is *n factorial*.]

End of Home-γ

γ**1:**	___ ___ ___	350pts
γ**2:**	___ ___ ___	350pts
Total:	___ ___ ___	700pts

HONOR CODE: *"I have neither requested nor received help on this exam other than from my team-mates and my professor (or his colleague)."* *Name/Signature/Ord*

Ord: ....

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