$\boldsymbol{\gamma}$ and $\boldsymbol{\Gamma}()$

Jonathan L.F. King
University of Florida, Gainesville FL 32611-2082, USA
squash@ufl.edu
Webpage http://squash.1gainesville.com/
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EM-number

Recall that the ***harmonic numbers*** $\mathsf{H}_N := \sum_{k=1}^{N} \frac{1}{k}$ upper/lower bound $\log()$, in that

$$\mathsf{H}_{N-1} \;>\; \log(N) \;>\; \mathsf{H}_N - 1\,.$$

The ***Euler-Mascheroni*** number[♡1] is defined as the asymptotic discrepancy,

1:
$$\begin{aligned}\boldsymbol{\gamma} \;&:=\; \lim_{N\nearrow\infty}\left[\mathsf{H}_{N-1} - \log(N)\right] \\ &= \lim_{N\nearrow\infty}\left[\mathsf{H}_N - \log(N)\right] \;=\; \int_1^\infty \Big[\frac{1}{\lfloor t\rfloor} - \frac{1}{t}\Big]\,\mathrm{d}t.\end{aligned}$$

Although estimate $\boldsymbol{\gamma} \approx 0.578^-$ does not help with rationality, it is reassuring to know.

1a: Lemma. *The N^{th} harmonic number satisfies*

$*1$:
$$\mathsf{H}_N \;=\; \sum_{k=1}^{\infty} \frac{N}{k[k+N]}\,.$$

Also

$*2$:
$$\frac{N}{k[k+N]} \;=\; \int_0^1 \frac{t^{k/N}}{k}\,\mathrm{d}t\,.$$

Consequently,

$*3$:
$$\mathsf{H}_N \;=\; -\int_0^1 \log(1 - t^{1/N})\,\mathrm{d}t\,. \qquad \diamondsuit$$

Pf $(*1)$. Sum $S_\ell := \sum_{k=1}^{\ell} \frac{N}{k[k+N]} = \sum_{k=1}^{\ell}\left[\frac{1}{k} - \frac{1}{k+N}\right]$ telescopes. Once $\ell \geq N$, then,

$$S_\ell \;=\; \Big[\sum_{k=1}^{N}\frac{1}{k}\Big] - \Big[\sum_{k=1}^{N}\frac{1}{\ell+k}\Big]\,.$$

The righthand-sum goes to zero, as $\ell\nearrow\infty$. ♦

Pf of $(*2)$. Observe $\int t^{\frac{k}{N}}\,\mathrm{d}t \;=\; \frac{N}{k+N} t^{\frac{k+N}{N}}$. Hence RhS$(*2)$ equals

$$\frac{N}{k[k+N]}\cdot t^{\frac{k+N}{N}}\Big|_{t=0}^{t=1} \;=\; \mathrm{LhS}(*2)\,. \qquad ♦$$

[♡1] As of Nov.2017, it is unknown if the ***EM-number*** $\boldsymbol{\gamma}$ is rational or irrational.

Pf of $(*3)$. One Taylor-series expansion for log is

$$-\log(1-x) \;\overset{\text{for } |x|<1}{=\!=}\; \sum_{k=1}^{\infty}\frac{x^k}{k}\,.$$

For $0 < t < 1$, then,

$$-\log(1 - t^{1/N}) \;=\; \sum_{k=1}^{\infty}\frac{t^{\frac{k}{N}}}{k}\,.$$

It is valid to commute $\int_0^1$ with the sum, as all the summands have the same sign. Thus

$$\begin{aligned}-\int_0^1 \log(1 - t^{1/N})\,\mathrm{d}t \;&=\; \sum_{k=1}^{\infty}\int_0^1 \frac{t^{\frac{k}{N}}}{k}\,\mathrm{d}t \\ &=\; \sum_{k=1}^{\infty}\frac{N}{k[k+N]} \;=\; \mathsf{H}_N\,. \qquad ♦\end{aligned}$$

1b: Lemma. *For all t with $0 < t < 1$,*

†:
$$\lim_{N\nearrow\infty} N\cdot\left[1 - t^{1/N}\right] \;=\; \text{-}\log(t)\,.$$

‡:
$$\lim_{N\nearrow\infty} \log(N\cdot[1 - t^{1/N}]) \;=\; \log(\text{-}\log(t))\,. \qquad \diamondsuit$$

Proof. Setting $h := \frac{1}{N}$, we can rewrite limit (†) as

$$-\lim_{h\searrow 0}\frac{t^h - 1}{h} \;\overset{\text{note}}{=\!=}\; -f'(0)\,,$$

where $f(h) := t^h$. By definition, $f(h) = \exp(\log(t)\cdot h)$. So $f'(h) = \log(t)\cdot f(h)$. Thus $f'(0) = \log(t)$.

Lastly, (‡) holds, since [the outer] log is continuous. ♦

1c: $\boldsymbol{\gamma}$-$\boldsymbol{\Gamma}$ Thm. *Using fnc $\boldsymbol{\Gamma}()$ from the next section,*

$$\boldsymbol{\gamma} \;=\; \int_0^1 \log(\text{-}\log(t))\,\mathrm{d}t \;=\; \text{-}\boldsymbol{\Gamma}'(1)\,. \qquad \diamondsuit$$

Proof. Eqns (1) and $(*3)$ say $\boldsymbol{\gamma}$ is the limit of

$$\begin{aligned}\mathsf{H}_N - \log(N) \;&=\; \Big[\text{-}\int_0^1 \log(1 - t^{1/N})\,\mathrm{d}t\Big] - \int_0^1 \log(N)\,\mathrm{d}t \\ &=\; \text{-}\int_0^1 \Big[\log(1 - t^{1/N}) + \log(N)\Big]\,\mathrm{d}t \\ &=\; \text{-}\int_0^1 \log\Big(N\cdot[1 - t^{1/N}]\Big)\,\mathrm{d}t\,.\end{aligned}$$

Skipping the justification needed to pass $\lim_{N\nearrow\infty}$ through the integral sign,

$$\gamma = -\int_0^1 \Big[\lim_{N\nearrow\infty} \log\Big(N\cdot[1 - t^{1/N}]\Big)\Big]\,dt$$

$$\stackrel{\text{by } (\ddagger)}{=\!=} -\int_0^1 \log(-\log(t))\,dt\,.$$

Exponential CoV. To compute this last integral, we use CoV $u = -\log(t)$. I.e, $t = e^{-u}$, so $\frac{dt}{du} = -e^{-u}$ and thus $dt = -e^{-u}\,du$.

As $0\nearrow t\nearrow 1$, remark that $\infty\searrow u\searrow 0$. Hence,

$$-\gamma = \int_0^1 \log(-\log(t))\,dt$$

1d:

$$\stackrel{\text{CoV}}{=\!=} \int_\infty^0 \log(u)\,\overbrace{[-e^{-u}]\,du}^{dt} = \int_0^\infty \log(u)\;e^{-u}\,du\,.$$

And this last equals $\Gamma'(1)$, courtesy (3) ♦

Gamma function

The ***Gamma fnc*** arises in volumes of N-dimensional balls, and in Laplace transforms. Leaving motivation for later, define the Gamma fnc by

2: $$\Gamma(z) := \int_0^\infty t^{z-1}\,e^{-t}\,dt\,, \quad \textit{for } z\in\mathbb{C} \textit{ with } \operatorname{Re}(z)>0\,.$$

When $\operatorname{Re}(w)>0$, note $\lim_{t\searrow 0}[t^w e^{-t}]$ is zero. Thus

$$t^w[-e^{-t}]\Big|_{t=0}^{t=\infty} = 0\,.$$

For $\operatorname{Re}(z)>1$, then, integration by parts produces

$$\Gamma(z) = t^{z-1}\cdot[-e^{-t}]\Big|_{t=0}^{t=\infty} - \int_0^\infty [z-1]t^{z-2}\cdot[-e^{-t}]\,dt. \;\; \textit{So}$$

2a: $$\Gamma(z) = [z-1]\cdot\Gamma(z-1)\,, \quad \textit{for } \operatorname{Re}(z)>1.$$

Since $\Gamma(1) = 1$,

2b: $$\Gamma(n) = [n-1]!\,, \quad \textit{for } n\in\mathbb{Z}_+.$$

As a consequence, binomial/multinomial coefficients can be generalized using Γ, and many of the identities extend.

Analytic continuation. Writing $w := z-1$,

2c: $$\Gamma(w) = \frac{1}{w}\cdot\Gamma(w+1)$$

from (2a). As (3) will show, on the righthand half-plane our $\Gamma()$ is ∞ly differentiable, hence is *analytic* courtesy the Cauchy-Goursat theorem.

Use (2c) to iteratively extend $\Gamma()$ to the complex plane. This $\Gamma()$ is analytic on $\mathbb{C}\smallsetminus\{0,-1,-2,-3,\dots\}$. Further, $\Gamma()$ is meromorphic on $\mathbb{C}$, with simple poles at $0,-1,-2,\dots$.

Why simple? Equality $\Gamma(1) = \int_0^\infty e^{-t}\,dt = 1$, together with (2c) imply that

$$\lim_{w\to 0} w\cdot\Gamma(w) = \lim_{w\to 0}\Gamma(w+1) = \Gamma(1) = 1\,.$$

This is not zero, hence $\Gamma()$ has a simple pole at 0. Then (2c) iteratively shows that the other poles are simple.

Residues. At $N = 0,1,2,\dots$, we compute the Γ-residue at $-N$. Iterating (2c),

$$*:\quad \Gamma(z) = \frac{1}{z}\cdot\frac{1}{z+1}\cdot\frac{1}{z+2}\cdots\frac{1}{z+N-1}\cdot\frac{1}{z+N}\cdot\Gamma(z+N+1)\,.$$

So $\operatorname{Res}_{z=-N}(\Gamma(z)) = \lim_{z\to -N}[z+N]\cdot\Gamma(z)$. By $(*)$, then,

2d: $$\operatorname{Res}_{z=-N}(\Gamma(z)) = \frac{[-1]^N}{N!}\,.$$

Calculus. Differentiating under the integral sign is valid in (2). For $k = 0,1,\dots$, applying $\frac{d}{dz}$ gives

3: $$\Gamma^{(k)}(z) = \int_0^\infty t^{z-1}e^{-t}\,[\log(t)]^k\,dt, \quad \textit{when } \operatorname{Re}(z)>0.$$

In particular, $$\Gamma'(1) = \int_0^\infty e^{-t}\log(t)\,dt \stackrel{\text{by (1d)}}{=\!=} -\gamma\,.$$

Lap. Use $\widehat{f}$ for the ***Laplace transform*** of f, where

$$\widehat{f}(s) = [\mathcal{L}(f)](s) := \int_0^\infty e^{-st}\cdot f(t)\cdot dt\,. \qquad \square$$

4: Theorem. *The Laplace transform of logarithm, for $s>0$, is*

$$\dagger: \quad \widehat{\log}(s) = \frac{1}{s}\left[\Gamma'(1) - \log(s)\right] = \frac{-1}{s}\left[\gamma + \log(s)\right].$$

Fix z with $\text{Re}(z) > -1$. *For $t>0$, define* $P(t) := t^z$. *Then*

$$\ddagger: \quad \widehat{P}(s) = \frac{\Gamma(z+1)}{s^{z+1}}, \quad \text{for } s > 0. \qquad \diamond$$

Pf of ($\dagger$). Fix an $s>0$. With $u := st$, then, $\mathrm{d}t = \frac{1}{s}\,\mathrm{d}u$. As $0 \nearrow t \nearrow \infty$, our $0 \nearrow u \nearrow \infty$, since $s>0$. Thus $\widehat{\log}(s)$ equals

$$\int_0^\infty \mathrm{e}^{-st}\log(t)\cdot \mathrm{d}t \stackrel{\text{CoV}}{=\!=} \int_0^\infty \mathrm{e}^{-u}\overbrace{\log(\tfrac{u}{s})}^{\log(t)}\cdot \overbrace{\tfrac{1}{s}\,\mathrm{d}u}^{\mathrm{d}t}. \quad \textit{So,}$$

$$s\cdot\widehat{\log}(s) = \int_0^\infty \mathrm{e}^{-u}[\log(u) - \log(s)]\,\mathrm{d}u = \Gamma'(1) - \log(s)\Gamma(1) = \Gamma'(1) - \log(s),$$

as claimed. ♦

Pf of ($\ddagger$). With $u := st$ as before, $\mathrm{d}t = \frac{1}{s}\,\mathrm{d}u$. So $\widehat{P}(s)$ equals

$$\int_0^\infty \mathrm{e}^{-st}\,P(t)\cdot \mathrm{d}t \stackrel{\text{since } s>0}{=\!=} \int_0^\infty \mathrm{e}^{-u}\,[\tfrac{u}{s}]^z\cdot \overbrace{\tfrac{1}{s}\,\mathrm{d}u}^{\mathrm{d}t} = \frac{1}{s^{z+1}}\int_0^\infty \mathrm{e}^{-u}\,u^z\,\mathrm{d}u. \qquad ♦$$

As our last fact in this introduction to $\Gamma()$, let's show that

$$5: \quad \Gamma(\tfrac{1}{2}) = \sqrt{\pi}.$$

We employ substitution $t = x^2$. Thus $\mathrm{d}t = 2x\,\mathrm{d}x$. As $0 \nearrow t \nearrow \infty$, note $0 \nearrow x \nearrow \infty$. Recall $\Gamma(\frac{1}{2})$ equals

$$\int_0^\infty \mathrm{e}^{-t}\,t^{\frac{1}{2}-1}\,\mathrm{d}t = \int_0^\infty \mathrm{e}^{-t}\,\tfrac{1}{t^{1/2}}\cdot \mathrm{d}t \stackrel{\text{CoV}}{=\!=} \int_0^\infty \mathrm{e}^{-[x^2]}\,\tfrac{1}{x}\cdot 2x\,\mathrm{d}x = 2\int_0^\infty \mathrm{e}^{-x^2}\,\mathrm{d}x = \int_{-\infty}^\infty \mathrm{e}^{-x^2}\,\mathrm{d}x.$$

This last integral equals $\sqrt{\pi}$, as shown in the next section by means of the famous Polar-Coordinate Trick.

Polar-coordinate Trick

Let $J := \int_{-\infty}^{+\infty}\mathrm{e}^{-x^2}\,\mathrm{d}x$. We use the ***PCT*** ("polar coordinate trick") to show that $J = \sqrt{\pi}$. We integrate the cartesian-square of the integrand to conclude that

$$6.1: \quad J^2 = \left[\int_{-\infty}^{+\infty}\mathrm{e}^{-[x^2]}\,\mathrm{d}x\right]\cdot\left[\int_{-\infty}^{+\infty}\mathrm{e}^{-[y^2]}\,\mathrm{d}y\right] = \int_{-\infty}^{+\infty}\mathrm{e}^{-[x^2+y^2]}\cdot \mathrm{d}(x,y) = \int_0^{+\infty}\mathrm{e}^{-r^2}\cdot \underbrace{2\pi r\cdot \mathrm{d}r}_{\substack{\text{Area of radius-}r\text{ annulus}\\ \text{of thickness } \mathrm{d}r.}}.$$

Hence $J^2 = \pi\cdot[-\mathrm{e}^{-r^2}]\big|_{r=0}^{r=+\infty} = \pi$. Since J is *the integral of a non-negative fnc*, nec. $J \geq 0$. Thus

$$6.2: \quad \int_{-\infty}^{+\infty}\mathrm{e}^{-x^2}\,\mathrm{d}x = \sqrt{\pi}.$$

Further Γ results

Commented-out ...