Distance between flats

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For a general field $\mathcal{F}$ and vectors $\mathbf{u}, \mathbf{w}$ in an $\mathcal{F}$-vector-space, define the "*line through* $\mathbf{u}$ *in direction* $\mathbf{w}$":

$$\operatorname{LinDir}(\mathbf{u}, \mathbf{w}) \;:=\; \{\mathbf{u} + t\mathbf{w} \mid t \in \mathcal{F}\}.$$

We now happily specialize to an IP (inner-product) space $\mathbf{V}$ over field $\mathcal{F} \subset \mathbb{C}$ which is sealed under complex-conjugation;

$$\forall \alpha \in \mathbb{C}: \quad \alpha \in \mathcal{F} \implies \overline{\alpha} \in \mathcal{F}.$$

Our IP is conjugate-linear in its 1st argument. I.e, for every $\beta \in \mathcal{F}$ and $\mathbf{u}, \mathbf{w} \in \mathbf{V}$:

$$\langle \beta\mathbf{u}, \mathbf{w}\rangle = \overline{\beta} \cdot \langle \mathbf{u}, \mathbf{w}\rangle \quad \text{and} \quad \langle \mathbf{u}, \beta\mathbf{w}\rangle = \beta \cdot \langle \mathbf{u}, \mathbf{w}\rangle.$$

Proj and Orth. For a direction-vector $\mathbf{D} \neq \vec{\mathbf{0}}$ and arbitrary $\mathbf{u} \in \mathbf{V}$, we define the orthogonal-projection operator:

$$\text{1:} \qquad \operatorname{Proj}_{\mathbf{D}}(\mathbf{u}) \;:=\; \frac{\langle \mathbf{D}, \mathbf{u}\rangle}{\langle \mathbf{D}, \mathbf{D}\rangle} \cdot \mathbf{D}.$$

Our IP is linear in its 2nd argument, so formula (1) indeed satisfies that $\operatorname{Proj}_{\mathbf{D}}(\beta\mathbf{u}) = \beta \cdot \operatorname{Proj}_{\mathbf{D}}(\mathbf{u})$.

This immediately gives that $\operatorname{Proj}_{\mathbf{D}}$ is idempotent: Write $\mathbf{w} := \operatorname{Proj}_{\mathbf{D}}(\mathbf{u}) = \alpha\mathbf{D}$, where $\alpha := \frac{\langle \mathbf{D}, \mathbf{u}\rangle}{\langle \mathbf{D}, \mathbf{D}\rangle}$. Then $\operatorname{Proj}_{\mathbf{D}}(\mathbf{w})$ equals $\alpha \cdot \operatorname{Proj}_{\mathbf{D}}(\mathbf{D}) = \alpha\mathbf{D} = \mathbf{w}$.

Let's also *check* that the difference $\mathbf{u} - \operatorname{Proj}_{\mathbf{D}}(\mathbf{u})$ *is* orthogonal to $\mathbf{D}$: Well, $\langle \mathbf{D}, \mathbf{u} - \operatorname{Proj}_{\mathbf{D}}(\mathbf{u})\rangle$ equals

$$\langle \mathbf{D}, \mathbf{u}\rangle - \langle \mathbf{D}, \mathbf{w}\rangle \;=\; \langle \mathbf{D}, \mathbf{u}\rangle - \alpha \cdot \langle \mathbf{D}, \mathbf{D}\rangle \;\stackrel{\text{note}}{=\!=}\; \vec{\mathbf{0}}.$$

Thus the formula for the orthogonal vector is indeed

$$\text{2:} \quad \operatorname{Orth}_{\mathbf{D}}(\mathbf{u}) \;:=\; \mathbf{u} - \operatorname{Proj}_{\mathbf{D}}(\mathbf{u}) \;=\; \frac{\langle \mathbf{D}, \mathbf{D}\rangle\mathbf{u} - \langle \mathbf{D}, \mathbf{u}\rangle\mathbf{D}}{\langle \mathbf{D}, \mathbf{D}\rangle}.$$

Let's compute the square-norms:

$$\text{1}': \qquad \|\operatorname{Proj}_{\mathbf{D}}(\mathbf{u})\|^2 \;=\; \frac{\langle \mathbf{D}, \mathbf{u}\rangle\langle \mathbf{u}, \mathbf{D}\rangle}{\langle \mathbf{D}, \mathbf{D}\rangle} \;=\; \frac{|\langle \mathbf{u}, \mathbf{D}\rangle|^2}{\langle \mathbf{D}, \mathbf{D}\rangle}.$$

$$\text{2}': \qquad \|\operatorname{Orth}_{\mathbf{D}}(\mathbf{u})\|^2 \;=\; \langle \mathbf{u}, \mathbf{u}\rangle - \|\operatorname{Proj}_{\mathbf{D}}(\mathbf{u})\|^2 \;=\; \frac{\langle \mathbf{u}, \mathbf{u}\rangle\langle \mathbf{D}, \mathbf{D}\rangle - \langle \mathbf{D}, \mathbf{u}\rangle\langle \mathbf{u}, \mathbf{D}\rangle}{\langle \mathbf{D}, \mathbf{D}\rangle}.$$

Hence the closest point on line $\operatorname{LinDir}(\mathbf{q}, \mathbf{D})$ to a point $\mathbf{p}$ is

$$\text{3:} \qquad \mathbf{q} + \operatorname{Proj}_{\mathbf{D}}(\mathbf{p} - \mathbf{q}).$$

The vector which goes from (3) to $\mathbf{p}$ is $\operatorname{Orth}_{\mathbf{D}}(\mathbf{p} - \mathbf{q})$.

Distance between two lines. Consider lines $\mathbb{L}_1 := \operatorname{LinDir}(\mathbf{p}_1, \mathbf{D})$ and $\mathbb{L}_2 := \operatorname{LinDir}(\mathbf{p}_2, \mathbf{E})$; the direction vectors $\mathbf{D}$ and $\mathbf{E}$ are not $\vec{\mathbf{0}}$.

Parallel Lines. WLOG, $\mathbf{D} = \mathbf{E}$. And $\operatorname{Dist}(\mathbb{L}_1, \mathbb{L}_2)$ equals $\|\operatorname{Orth}_{\mathbf{D}}(\mathbf{p}_1 - \mathbf{p}_2)\|$. Compute this via (2′).

Skew Lines. Now $\mathbf{D}, \mathbf{E}$ are *not* parallel. By Cauchy-Schwarz, then,

$$\text{4:} \qquad \langle \mathbf{D}, \mathbf{D}\rangle \cdot \langle \mathbf{E}, \mathbf{E}\rangle \;\neq\; \langle \mathbf{D}, \mathbf{E}\rangle \cdot \langle \mathbf{E}, \mathbf{D}\rangle.$$

(Indeed, difference $\langle \mathbf{D}, \mathbf{D}\rangle \cdot \langle \mathbf{E}, \mathbf{E}\rangle - \langle \mathbf{D}, \mathbf{E}\rangle \cdot \langle \mathbf{E}, \mathbf{D}\rangle$ is positive.)

Let's translate the lines by $-\mathbf{p}_1$. I.e, write them as $\operatorname{LinDir}(\vec{\mathbf{0}}, \mathbf{D})$ and $\operatorname{LinDir}(\mathbf{b}, \mathbf{E})$, where $\boxed{\mathbf{b} := \mathbf{p}_2 - \mathbf{p}_1}$.

Take a "moving point" on each line. I.e, for "times" $s, t \in \mathcal{F}$:

$$\text{5:} \qquad \begin{aligned} \mathbf{q}(s) &:= s\mathbf{D}. \\ \mathbf{r}(t) &:= \mathbf{b} - t\mathbf{E}. \end{aligned}$$

Since $\mathbf{D}, \mathbf{E}$ are not parallel, there will be a unique pair (s, t) of times such that $\mathbf{q}(s) - \mathbf{r}(t)$ is orthogonal to each direction-vector.

Orthogonal to $\mathbf{D}$ means $\langle \mathbf{D}, s\mathbf{D}\rangle - \langle \mathbf{D}, \mathbf{b} - t\mathbf{E}\rangle$ is zero. This is the first eqn below:

$$\text{6:} \qquad \begin{aligned} \langle \mathbf{D}, \mathbf{D}\rangle s \;+\; \langle \mathbf{D}, \mathbf{E}\rangle t \;&=\; \langle \mathbf{D},\, \mathbf{b}\rangle. \\ \langle \mathbf{E}, \mathbf{D}\rangle s \;+\; \langle \mathbf{E}, \mathbf{E}\rangle t \;&=\; \langle \mathbf{E},\, \mathbf{b}\rangle. \end{aligned}$$

Orthogonality to $\mathbf{E}$ yields the 2nd eqn. Courtesy (4), the coefficient-matrix

$$\mathsf{M} \;:=\; \begin{bmatrix} \langle \mathbf{D}, \mathbf{D}\rangle & \langle \mathbf{D}, \mathbf{E}\rangle \\ \langle \mathbf{E}, \mathbf{D}\rangle & \langle \mathbf{E}, \mathbf{E}\rangle \end{bmatrix}$$

is non-singular. Let $R := 1/\operatorname{Det}(\mathsf{M})$ be the reciprocal.

The unique solution $(s, t) := (\sigma, \tau)$ to (6) is thus

$$\text{7:} \qquad \begin{bmatrix} \sigma \\ \tau \end{bmatrix} \;=\; R \cdot \begin{bmatrix} \langle \mathbf{E}, \mathbf{E}\rangle & -\langle \mathbf{D}, \mathbf{E}\rangle \\ -\langle \mathbf{E}, \mathbf{D}\rangle & \langle \mathbf{D}, \mathbf{D}\rangle \end{bmatrix} \cdot \begin{bmatrix} \langle \mathbf{D},\, \mathbf{b}\rangle \\ \langle \mathbf{E},\, \mathbf{b}\rangle \end{bmatrix}.$$

Plugging this (σ, τ) pair into (5) gives us the unique pair of closest points on the translated lines. So the closest points[♡1] on the original lines are

$$\text{8:} \qquad \begin{aligned} \mathbb{L}_1 \;\ni\; \mathbf{p}_1 + \mathbf{q}(\sigma) \;&=\; \mathbf{p}_1 + \sigma\mathbf{D} \qquad \text{and} \\ \mathbb{L}_2 \;\ni\; \mathbf{p}_1 + \mathbf{r}(\tau) \;&=\; \mathbf{p}_2 - \tau\mathbf{E}. \end{aligned}$$

[♡1] The asymmetry in (8) comes from the asymmetry in $\mathbf{b}$. We had a choice between $[\mathbf{p}_2 - \mathbf{p}_1]$ and $[\mathbf{p}_1 - \mathbf{p}_2]$.

Gram-Schmidt alg. In real-or-complex IPS $\mathbf{H}$, consider vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots$. We will construct pairwise-orthogonal non-$\vec{\mathbf{0}}$ vectors $\mathbf{b}_1, \mathbf{b}_2, \ldots$ and natnums $K_0{=}0 < K_1 < K_2 < \ldots$ so that this holds:

For each $n = 0, 1, 2, \ldots$:
Subspace $\mathbf{W}_n \coloneqq \operatorname{Spn}(\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\})$
equals $\operatorname{Spn}(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{b}_{K_n}\})$.

For $n = 0$, our $\mathbf{W}_0$ equals $\{\vec{\mathbf{0}}\}$, which indeed is the span of taking none of the $\mathbf{v}$-vectors.

At stage N: Successively set κ to the indices *after* K_N. For each, compute

$$\mathbf{g}_\kappa \coloneqq \operatorname{Orth}_{\mathbf{W}_N}(\mathbf{v}_\kappa) \stackrel{\text{note}}{=\!=} \mathbf{v}_\kappa - \sum_{j=1}^{N} \operatorname{Proj}_{\mathbf{b}_j}(\mathbf{v}_\kappa)\,,$$

stopping at the *first* κ where $\mathbf{g}_\kappa \neq \vec{\mathbf{0}}$. Set $\mathbf{b}_{N+1} \coloneqq \mathbf{g}_\kappa$ and $K_{N+1} \coloneqq \kappa$. Now increment N.

Cauchy-Schwarz Inequality. Two vectors $\mathbf{v}, \mathbf{w}$ in an $\mathcal{F}$-VS are ***$\mathcal{F}$-parallel*** if, there exists scalars $\alpha, \beta \in \mathcal{F}$, not both 0, with $\alpha\mathbf{v} = \beta\mathbf{w}$; i.e, if $\{\mathbf{v}, \mathbf{w}\}$ is lin-dependent.

Now let $\mathcal{F}$ be a subfield of $\mathbb{C}$ and consider relation $\alpha\mathbf{v} = \beta\mathbf{w}$, for real α,β, not both 0. If we can pick α,$\beta \geq 0$, then $\mathbf{v}$ and $\mathbf{w}$ are ***same-direction parallel***. If we can pick $\alpha \leq 0 \leq \beta$, then $\mathbf{v}$ and $\mathbf{w}$ are ***opposite-direction parallel***. (Abbrev: "same-dir parallel" and "opp-dir parallel".)

Remark. None of *same/opp-dir parallel* is transitive. However, let $\mathcal{R}$ be any one of the three relations. Then

If $\mathbf{v}\mathcal{R}\mathbf{c}$ *and* $\mathbf{c}\mathcal{R}\mathbf{w}$, *and* $\mathbf{c} \neq \vec{\mathbf{0}}$, *then* $\mathbf{v}\mathcal{R}\mathbf{w}$. □

9: Cauchy-Schwarz Theorem. *Consider vectors* $\mathbf{v}, \mathbf{w}$ *in a complex IPS* $\mathbf{V}$. *Then*

9a: $$\left|\langle \mathbf{v}, \mathbf{w}\rangle\right|^2 \quad\leq\quad \|\mathbf{v}\|^2 \cdot \|\mathbf{w}\|^2\,,$$

with equality IFF $\mathbf{v}$ *and* $\mathbf{w}$ *are parallel. Indeed* $\langle \mathbf{v}, \mathbf{w}\rangle = \pm\|\mathbf{v}\| \cdot \|\mathbf{w}\|$ *as the* $\mathbf{v}, \mathbf{w}$ *pair is same/opp-dir parallel.* ◊

9b: Lem. *Consider quadratic* $h(t) \coloneqq At^2 - 2Bt + C$, *where* A,B,C *are real, with* $A > 0$. *Let* $\boldsymbol{\tau} \in \mathbb{R}$ *be the min-point of* h, *i.e,* $[\forall t \in \mathbb{R}\colon\ h(t) \geq h(\boldsymbol{\tau})]$. *Then*

9c: $\boldsymbol{\tau} = \frac{B}{A}$, *and min-value* $h(\boldsymbol{\tau})$ *equals* $C - \frac{B^2}{A}$. ◊

Pf. Well, $h'(t) = 2At - 2B$, and $h'(\boldsymbol{\tau}) = 0$. Etc. ♦

Pf of C-S, (9), for a $\mathbb{R}$-VS. If $\mathbf{w} = \vec{\mathbf{0}}$, then the thm's conclusion holds, so WLOG $\mathbf{w} \neq \vec{\mathbf{0}}$.

Define $f(t) \coloneqq \|\mathbf{v} - t\mathbf{w}\|^2$. Courtesy bilinearity,

$$f(t) \quad=\quad At^2 - 2Bt + C\,,$$

where $A \coloneqq \langle \mathbf{w}, \mathbf{w}\rangle$, $C \coloneqq \langle \mathbf{v}, \mathbf{v}\rangle$ and $B \coloneqq \langle \mathbf{v}, \mathbf{w}\rangle$. Since f is the square of a distance, the min-value of f is non-negative. Thus $0 \leq C - \frac{B^2}{A}$. But $A > 0$, so $\boxed{B^2 \leq AC}$. And this is (9a). ♦

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