The Ben Problems

Autumn 2015 LinAlg class
Webpage http://squash.1gainesville.com/course.DfyQ.2015t.html
16 March, 2016 (at *00:33*)

Standing notation. Symbols $\mathbf{V}, \mathbf{W}, \mathbf{H}$ denote vectorspaces, unless stated otherwise.

To indicate that sets A, B are disjoint, I may write $A \sqcap B$ rather than $A \cap B = \varnothing$. A pair of sets $\mathcal{A}, \mathcal{B} \subset \mathbf{V}$ is ***span-disjoint*** if

$$\mathcal{A} \sqcap \mathrm{Spn}(\mathcal{B}) \quad \textit{and} \quad \mathrm{Spn}(\mathcal{A}) \sqcap \mathcal{B}.$$

Use VS to abbreviate 'vectorspace', and VSS abbreviates 'VectorSubSpace'. □

Ben1: "Push-forwards of VSSes are VSSes."

Consider a linear map $\mathsf{T}: \mathbf{V} \to \mathbf{H}$, and $\mathbf{W} \overset{\text{VSS}}{\subset} \mathbf{V}$. Prove that $\mathsf{T}(\mathbf{W})$ is a VSS of $\mathbf{H}$.

Ben2: Consider a span-disjoint pair $\mathcal{A}, \mathcal{B} \subset \mathbf{V}$ such that $\mathcal{A}$ is L.I and $\mathcal{B}$ is L.I, and each set is finite. Prove, or provide a CEX: *Union $\mathcal{A} \cup \mathcal{B}$ is L.I.*

[In class we proved that when $\mathcal{B} = \{\mathbf{v}\}$ is a singleton, then "yes", union $\mathcal{A} \cup \{\mathbf{v}\}$ is L.I.]

Ben3: Collection $\mathcal{Y} \subset \mathrm{MAT}_{3\times 3}(\mathbb{R})$, where

$$\mathcal{Y} \quad := \quad \left\{ \mathbf{v}^{\mathsf{t}} \cdot \mathbf{w} \;\middle|\; \mathbf{v}, \mathbf{w} \in \mathrm{MAT}_{1\times 3} \right\}.$$

Since $\mathrm{MAT}_{3\times 3}$ has 9 degrees-of-freedom whereas $\mathcal{Y}$ has only 6, our $\mathcal{Y}$ is *not* all of $\mathrm{MAT}_{3\times 3}$. Prove, or disprove: *Our $\mathcal{Y}$ is a VSS of $\mathrm{MAT}_{3\times 3}$.*

Ben4: Evidently $\mathcal{W} \subset \mathrm{MAT}_{2\times 2}$, where

$$\mathcal{W} \quad := \quad \left\{ \mathbf{v}^{\mathsf{t}} \cdot \mathbf{w} \;\middle|\; \mathbf{v}, \mathbf{w} \in \mathrm{MAT}_{1\times 2} \right\}.$$

Prove, or provide a CEX: *This $\mathcal{W}$ is all of $\mathrm{MAT}_{2\times 2}$.*

Ben5: Does $\mathcal{Y} := \left\{ \mathbf{v}^{\mathsf{t}} \cdot \mathbf{w} \;\middle|\; \mathbf{v}, \mathbf{w} \in \mathrm{MAT}_{1\times 3} \right\}$ span $\mathrm{MAT}_{3\times 3}$? I.e, does $\mathrm{Spn}(\mathcal{Y}) \overset{?}{=} \mathrm{MAT}_{3\times 3}$?