Topo 2: MTG4303 and MTG5317 **Home-E** Prof. JLF King *Touch*: 23Sep2017

"*Finishing the second Semester in Style*"

Due: Thursday, **23Apr1998** by 4PM, slid under my office door.

Notation. TS, MS, topological/metric space. BCT, Baire Category Thm. LCG, Locally countably generated. (LCG means: *Each point has a countable local-base.* The standard, but non-intuitive, term is "*first-countable*".)

Suppose Y,X are TSes, with $Y \subset X$. Then Y is a ***subspace*** of X, if Y's topology is indeed that which is induced from X.

E1: The indicator function $\mathbf{1}_{\mathbb{Q}}$ is discontinuous everywhere.

a Use BCT on $\mathbb{R}$ to show that $\mathbf{1}_{\mathbb{Q}}$ is **not** the pointwise limit of a sequence of continuous functions.

b In contrast, produce a *net* $(f_i)_{i \in D}$ of continuous functions such that

$$f_i \rightarrowtail \mathbf{1}_{\mathbb{Q}}$$

[*Hint:* Given finitely many reals $(y_j)_{j=1}^N$ and $x_1 < x_2 < \cdots < x_N$, it is convenient to prove a LEMMA: *There is a continuous function f on $\mathbb{R}$ such that $\forall j$: $f(x_j) = y_j$.*]

E2: Let $\Omega := \mathbb{R} \times [0, \infty)$ be the upper half-plane, equipped with the Tangent-Ball Topology of 3.5 of §C of my notes. Its topology, $\mathcal{D}$, is generated by sets of these forms, for reals x and $y \geqslant r > 0$:

$$\begin{aligned} &\mathrm{Bal}_r((x,y)), && \text{usual balls;} \\ W_r(x) \;:=\; &\{(x,0)\} \cup \mathrm{Bal}_r((x,r)), && \text{weird balls.} \end{aligned}$$

We showed in class (Thanks Suzanne!) that $\mathcal{D}$ is regular. Prove that $\mathcal{D}$ is **not** normal, using these steps:

i Let $X \subset \Omega$ denote the x-axis. Show that each subset $S \subset X$ is $\mathcal{D}$-closed (i.e, regarded as a subset of Ω).

ii Write $X = \mathbb{Q} \sqcup I$, where I is the set of irrationals on the x-axis. From part (i), each of $\mathbb{Q}$ and I is $\mathcal{D}$-closed. Now suppose that U and J are $\mathcal{D}$-open sets such that $U \supset \mathbb{Q}$ and $J \supset I$. Prove that U must intersect J, as follows:

Let $M_k \subset I$ be the set of $x \in I$ such that

$$W_{1/k}(x) \;\subset\; J\,,$$

and let $\widetilde{M_k}$ denote its closure in $\mathbb{R}$ under the *standard* topology. Argue geometrically that if $U \sqcap J$, then $\widetilde{M_k}$ is $\mathbb{R}$-meager. (This is the key step —be *careful* and *precise*!)

Now use BCT on $\mathbb{R}$ to carefully get a contradiction.

E3: With X a normal TS which is not compact, let $\widehat{X}$ denote its Stone-Čech compactification. Prove that $\widehat{X}$ is not LCG. (In particular, $\widehat{X}$ is not metrizable.) [*Hint:* See Munkres #9 on P.243.]

E4: Carefully do the retraction problem, Munkres #**3**P.*330*. This problem is all "definition chasing", but will require you to read ahead.

E5: Invent a good, interesting problem which involves either BCT or Stone-Čech or Urysohn/Tietze or space-filling curves or manifolds. Preferably –but not necessarily– give a solution to your problem.

Alternatively, generalize one of the exam problems in some interesting way.

Extra problems from a make-up exam

ES1: In a Hausdorff TS X, consider a subset E. For each of the following, give a *proof* or *counterexample*.

a Suppose E is connected. Then its boundary, ∂E, is connected.

b If ∂E is connected then so is E.

c If E is connected then its interior, E°, is connected.

d If E° is connected, then so is E.

ES2: A TS is ***nifty*** if: *Each closed subset is a $\mathcal{G}_\delta$ set.*

i Let $\mathcal{D}$ be the co-finite topology on $\mathbb{R}$. Show that TS $(\mathbb{R}, \mathcal{D})$ is <u>not</u> nifty.

ii Prove that each MS <u>is</u> nifty.

iii Give an example –with proof!– of a compact Hausdorff space which is *not* nifty.

ES3: Let $\mathbb{R}_n$ be a copy of $\mathbb{R}$, and let $\mathcal{U}_n$ denote the usual topology on $\mathbb{R}_n$. Let $\mathcal{T}$ be the box topology on

$$\Omega \quad := \quad \mathbb{R}_1 \times \mathbb{R}_2 \times \ldots.$$

Prove that $(\Omega, \mathcal{T})$ is a ***BaireCat*** space, i.e, every $\mathcal{T}$-residual set is dense.

ES4: Suppose Z is a normal TS which is not compact. Let $\widehat{Z}$ denote its Stone-Čech compactification. Prove that $\widehat{Z}$ is not LCG. (In particular, $\widehat{Z}$ is not metrizable.) [*Hint:* See Munkres **#9**P *243*.]

ES5: Let $Y_1 \subset Y_2 \subset \ldots$ be TSes, where each Y_n is a closed *subspace* of Y_{n+1}. Let $\Lambda := \bigcup_{n=1}^{\infty} Y_n$, and say that a subset $U \subset \Lambda$ is ***green*** IFF: For each n the intersection $U \cap Y_n$ is Y_n-open.

a Show that the green sets form a topology on Λ.

b We henceforth equip Λ with the green topology. Show that Y_1 is a Λ-*closed* subspace of Λ. What about Y_n?

c [The real question.] Now assume that each Y_n is a *normal* TS. Prove that Λ is then also normal.

End Notes and Hints. For(ES1), different counterexamples may use different TSes X and different sets E.

For (ES3), since Ω is not locally-compact nor metrizable, it does not satisfy the hypotheses of BCT. Nonetheless, you can mimic the proof of BCT, keeping track of what happens in each $\mathbb{R}_n$ component.

Important: Be unambiguous about which topology you are using, e.g, "…is $\mathcal{U}_3$-open", "Take the $\mathcal{T}$-closure of…", "…is $\mathcal{U}_n$-residual" etc.

Here is one approach to (ES5). We need –given disjoint Λ-closed subsets A and B– to produce disjoint Λ-*open* sets $U \supset A$ and $W \supset B$. Use the Tietze Extension Theorem to argue that you can build functions $(h_n)_{n=1}^{\infty}$ so that

I: $h_n{:}Y_n \to [0,1]$ and is Y_n-continuous;

II: Its restrictions satisfy $h_n\lfloor_A \equiv 0$ and $h_n\lfloor_B \equiv 1$;

III: h_n extends h_{n-1}.

(For $n = 0$, let Y_0 be the emptyset, and let h_0 be the void function.) Argue that the h_n functions can be stitched together to make a Λ-continuous function $g{:}\Lambda \to [0,1]$, then make use of this function. □

This was commented-out. Let's restate (ES5). Say that A,B is a ***good-pair*** if they are Λ-closed disjoint subsets of Λ. The pair is ***nice*** if there exist disjoint Λ-open sets $U \supset A$ and $W \supset B$. You need to show that every good-pair is nice.

Here is one approach. Let $A_n := A \cap Y_n$ and $B_n := B \cap Y_n$. Argue that each pair A_n,B_n is a good-pair. Argue that if each pair A_n,B_n is nice, then so is pair A,B.

Lastly, cleverly use the Tietze extension theorem to prove that each pair A_n,B_n is in fact nice. (This is the step to be careful on. Since Λ is not yet known to be normal, you can not apply Tietze to Λ.) □

End of Home-E