ACT
MAA4211 7222

**Home-E**

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*Carefully* TYPE *a triple–spaced, grammatical, essay solving the problem. I suggest* LaTeX*; you should learn it now, since it will be* **required** *next semester. This final-project is due by* **2PM, Friday, 12Dec2008**, *slid completely under my office door,* LIT402.

*Fill-in* every *blank* on this sheet.

Essays violate the CHECKLIST at *Grade Peril*...

**E1:** For $N{=}1,2,\ldots$, define fncs $R_N, F_N{:}\mathbb{R}{\to}\mathbb{R}$ by

$$R_N(x) := \exp(\tfrac{N}{1+Nx^2}) \quad\text{and}\quad F_N := 1/R_N\,.$$

**a** Prove that, pointwise, $F_1 \geqslant F_2 \geqslant F_3 \geqslant \ldots \geqslant 0$. Use this to prove, $\forall x \in \mathbb{R}$, that $L(x) := \lim_{n\to\infty} F_n(x)$ exists in $\mathbb{R}$. Computing, $L(0) =$ ________.

**b** There is a "Calc 1" fnc $M(x) :=$ ________________, defined[♡1] on $U := \mathbb{R} \smallsetminus \{0\}$, so that $L\!\downharpoonright_U = M\!\downharpoonright_U$. Differentiating, $M'(x) = M(x)\cdot\Big[$ ____________ $\Big]$.

Graph $L$ and some of the $F_n$. Indicate horizontal asymptotes, points of inflexion, local max/min-points, TW9Ys.[♡2]

Let $\|\cdot\|$ denote the supremum-norm on functions. Compute $\|L - F_n\|$. *Prove* or *dis*prove that

$$F_n \xrightarrow[n\to\infty]{\text{uniformly}} L. \quad (?)$$

Is the $(F_n)_{n=1}^{\infty}$ sequence $\|\cdot\|$-Cauchy?

**c** In any case, prove that $L()$ is continuous at zero. Please prove the.fol proposition, which may help with the above and below tasks. (Is L'H useful?)

**1: Prop'n.** *For each polynomial $P()$, necessarily $\lim_{x\to 0}\big[P(\frac{1}{x})\cdot M(x)\big]$ equals zero.* ◊

With this tool in hand, prove that $L$ is differentiable at zero by directly applying the **definition** (difference-quotient) of derivative.

**d** Now prove that $L$ is $\infty$ly differentiable by establishing the following nifty lemma.

**2: Nifty Poly Lemma.** *Given a polynomial $P$, define*

$$\textbf{3:}\qquad L_P\Big(x\Big) \;:=\; \begin{Bmatrix} 0\,, & \text{if } x=0 \\ P(\frac{1}{x})\cdot M(x)\,, & \text{if } x\neq 0 \end{Bmatrix}.$$

*Prove that $L_P$ is differentiable at the origin (use the* **defn** *of derivative), as well as everywhere else. Give a formula (in terms of $P$) for a new polynomial $\widehat{P}$ satisfying that* $\boxed{[L_P]' = L_{\widehat{P}}}$. *(The foregoing* **Proposition** *may help in proving equality at zero.)* ◊

At this juncture, let $Q_0$ be the constant-1 polynomial. Inductively define $Q_{k+1} := \widehat{Q_k}$, for each $k \in \mathbb{N}$. Prove that $L$ is $\infty$ly differentiable by showing that $L^{(k)} = L_{Q_k}$, for each $k = 0,1,2,\ldots$.

Exhibit a table showing the coefficients of polynomial $Q_k$, for $k = 1,2,3,4$, at least.

(Do you see a pattern, or a recurring property, here?)

**e** Compute the Taylor series, centered at zero, for $L()$. It has form $\sum_{k=0}^{\infty} C_k \cdot [x-0]^k$ where $C_k =$ ____________.

What is the RoC (radius-of-convergence) of this Taylor series?

What surprising relation do you note between $L()$ and its Taylor series?

| | | |
|---|---|---|
| **E1:** | ___ ___ | 55pts |
| *Poorly stapled, or missing ordinal* : | ___ | −5pts |
| *Missing name, or honor signature* : | ___ | −5pts |
| *Poorly proofread*: | ___ ___ ___ | −One Million pts |
| ***Total***: | ___ ___ ___ | ±55pts |

Print name ________________________________ Ord:

HONOR CODE: *"I have neither requested nor received help on this exam other than from my professor."*

Signature: ________________________________

[♡1] Note that $L()$ and $M()$ are *different* functions; after all, they have different domains.

[♡2] The Whole Nine Yards.