Existence and uniqueness of solution to an ODE

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ABSTRACT: Exposition of the standard contraction-mapping proof for a 1st-order DE. Shows how to apply this to higher-order DEs.

(The Fundamental Thm of ODEs, abbreviated FTODE: Wikipedia calls this either Picard-Lindelöf thm or Picard's existence thm or Cauchy-Lipschitz thm.)

Entrance. Consider[♡1] an initial-condition-DE

1: $$\mathbf{x}'(t) = \mathbf{K}\big(t;\, \mathbf{x}(t)\big), \quad \text{where } \mathbf{x}(5) = \mathbf{0}.$$

Here $\mathbf{x}$ is the *unknown function*. Putting sufficient conditions on $\mathbf{K}()$, the ***kernel***, will guarantee a local solution; a *unique* local soln.

The Setting

Fix a Banach space $\mathbb{H}$ with $\mathbf{0}$ its zero vector; use $\lceil\cdot\rceil$ for its norm. Let $\mathbf{B} \subset \mathbb{H}$ be a *closed* ball centered at $\mathbf{0}$, with Radius($\mathbf{B}$) > 0. (Possibly $\mathbf{B}$ will have infinite-radius and be all of $\mathbb{H}$; in that instance, step (6), below, is superflous, and consequently condition (A2) is unnecessary.) Typically, $\mathbb{H}$ is a finite-dim'al Euclidean space $\mathbb{R}^{\times N}$. Then, if Radius($\mathbf{B}$) *is* finite, our $\mathbf{B}$ will automatically be compact.

The "contraction space" Ω. Consider a compact interval

$$J_w := [5-w,\ 5+w],$$

where posreal w is what we'll call its **"width"**. Let Ω_w be the space of **continuous** functions

$$\mathbf{z}\colon J_w \to \mathbf{B} \quad \textit{such that} \quad \mathbf{z}(5) = \mathbf{0}.$$

Our goal is to find a differentiable fnc $\mathbf{x} \in \Omega_w$ fulfilling (1) for all $t \in J_w$. Henceforth write these two sets as J and Ω; the dependency on the width was made explicit, above, since *we will later shrink* w. Use $[\![\cdot]\!]$ to indicate the sup-norm,

2a: $$[\![\mathbf{z}]\!] := \sup_{t\in J} \lceil \mathbf{z}(t) \rceil,$$

for functions mapping $J \to \mathbb{H}$. For each $\mathbf{z} \in \Omega$, since J is compact, $[\![\mathbf{z}]\!] < \infty$.

Convergence in $[\![\cdot]\!]$ is uniform-convergence. The usual "three-ε argument" shows that the uniform-limit of continuous fncs (exists and) is cts. Thus

2b: *Metric space* $(\Omega,\ [\![\cdot]\!])$ *is complete.*

(This used that $\mathbf{B}$ is closed.) Each contraction mapping $\Omega \to \Omega$ has a (unique) fixed-point. Our goal:

2c: *Produce a contraction mapping* $\mathbf{z} \mapsto \widehat{\mathbf{z}}$ *whose unique fixed-point is a soln to* (1).

Conditions on K

Our kernel $\mathbf{K}$ is a function

A0: $$\mathbf{K}\colon J \times \mathbf{B} \to \mathbb{H}$$

satisfying these three conditions:

A1: $\mathbf{K}()$ *is continuous.*

A2: $\mathbf{K}()$ *is bounded. I.e, its sup-norm* $\|\mathbf{K}\|_{J\times\mathbf{B}}$ *is finite.* [*This boundedness is* *<u>not</u>* *needed if* $\mathbf{B}$ *is all of* $\mathbb{H}$.]

A3: $\mathbf{K}()$ *is* $\mathbb{H}$*-wise Lipschitz.*

This last means we have a real number $\mathcal{U} < \infty$, where $\mathcal{U}$ is the supremum of

$$\frac{\lceil \mathbf{K}(t;\,\mathbf{z}) - \mathbf{K}(t;\,\mathbf{y}) \rceil}{\lceil \mathbf{z} - \mathbf{y} \rceil}$$

[♡1] **Phrases:** WLOG: '*Without loss of generality*'. TFAE: '*The following are equivalent*'. ITOf: '*In Terms Of*'. OTForm: '*of the form*'. FTSOC: '*For the sake of contradiction*'. Use iff: '*if and only if*'.
IST: '*It Suffices to*' as in ISTShow, ISTExhibit.
Use w.r.t: '*with respect to*' and s.t: '*such that*'.
Latin: e.g: *exempli gratia*, '*for example*'. i.e: *id est*, '*that is*'. N.B: *Nota bene*, '*Note well*'. QED: *quod erat demonstrandum*, meaning "end of proof".

taken over all $t \in J$ and all distinct-point pairs $\mathbf{z}, \mathbf{y} \in \mathbf{B}$. In particular

$$\underset{\forall \mathbf{z},\mathbf{y}\in\mathbf{B}}{\forall t\in J,}: \ \lceil \mathbf{K}(t;\,\mathbf{z}) - \mathbf{K}(t;\,\mathbf{y})\rceil \ \leqslant\ \mathcal{U}\cdot\lceil \mathbf{z}-\mathbf{y}\rceil\,.$$

Evidently

A4: *If* $\mathbf{B}$ *is compact then* $J\times\mathbf{B}$ *is cpt, hence* (A2). *And automatically* (A3), *when* $\mathbf{B}$ *is compact and the partial derivative* $\frac{\mathrm{d}\mathbf{K}}{\mathrm{d}\mathbf{y}}(t;\,\mathbf{y})$ *exists, and is a continuous function of* $(t, \mathbf{y}) \in J\times\mathbf{B}$.

Each *function* $\mathbf{z}\in\Omega$ has a **"K-*image*"**, which is a fnc $f_{\mathbf{z}}: J\to\mathbb{H}$. It is

4a: $$f_{\mathbf{z}}(t) \ :=\ \mathbf{K}\Big(t;\,\mathbf{z}(t)\Big)\,.$$

Our Lipschitz constant $\mathcal{U}$ yields this:

For all functions $\mathbf{z}, \mathbf{y} \in \Omega$*:*

4b: $$[\![f_{\mathbf{z}} - f_{\mathbf{y}}]\!] \ \leqslant\ \mathcal{U}\cdot[\![\mathbf{z}-\mathbf{y}]\!]\,.$$

The operator

Each $f_{\mathbf{z}}$ is cts, courtesy the cty of $\mathbf{K}$. Thus (1), with DE $\boxed{\mathbf{x}'=f_{\mathbf{x}}}$, is equivalent to assertion

1′: $$\mathbf{x}(t) \overset{?}{=} \int_5^t f_{\mathbf{x}} \overset{\text{def}}{=} \int_5^t f_{\mathbf{x}}(\tau)\cdot\mathrm{d}\tau\,, \quad \text{for every time } t\in J.$$

This, courtesy the Fund.thm of Calculus, since $f_{\mathbf{x}}$ is cts so RhS(**??**′) is defined and differentiable.

Each fnc $\mathbf{z}\in\Omega$ yields a new fnc $\widehat{\mathbf{z}}: J\to\mathbb{H}$,

5: $$\widehat{\mathbf{z}} \ :=\ \left[t \ \mapsto\ \int_5^t f_{\mathbf{z}}\right].$$

A solution $\mathbf{x}$ to (**??**′) is a **fixed-point** of the mapping $\mathbf{z}\mapsto\widehat{\mathbf{z}}$. Each $\widehat{\mathbf{z}}$ maps into $\mathbb{H}$; as a first step, we need to guarantee that $\widehat{\mathbf{z}}$ maps into $\mathbf{B}$. So we need to arrange that the norm of $\widehat{\mathbf{z}}$ is $\leqslant$Radius($\mathbf{B}$). From (5) we have, since w is the radius of J, that

$$[\![\widehat{\mathbf{z}}]\!] \ \leqslant\ w\cdot[\![f_{\mathbf{z}}]\!] \ \overset{\text{note}}{\leqslant}\ w\cdot\|\mathbf{K}\|_{J\times\mathbf{B}}\,.$$

Courtesy (A2), our $\mathbf{K}()$ is bounded. So we can simply shrink[♡1] width w until the product

6: $w\cdot\|\mathbf{K}\|$ *is dominated by* Radius($\mathbf{B}$).

This arranges that the $[\mathbf{z}\mapsto\widehat{\mathbf{z}}]$ mapping indeed maps Ω into Ω.

[♡1]Shrinking w shrinks interval J and so might decrease norm $\|\mathbf{K}\|_{J\times\mathbf{B}}$. Lipschitz constant $\mathcal{U}$ might also decrease.

The Contraction

For each two functions $\mathbf{z},\mathbf{y}\in\Omega$ observe that

7: $$\begin{aligned}[\![\widehat{\mathbf{z}}-\widehat{\mathbf{y}}]\!] \ &\leqslant\ w\cdot[\![f_{\mathbf{z}}-f_{\mathbf{y}}]\!] \\ &\leqslant\ w\cdot\mathcal{U}\cdot[\![\mathbf{z}-\mathbf{y}]\!]\,.\end{aligned}$$

Again[♡1] shrink w, this time so that

6′: $$w\cdot\mathcal{U}\ <\ 1\,.$$

Now, finally, $\mathbf{z}\mapsto\widehat{\mathbf{z}}$ is a contraction-map on Ω and thus has a unique fixed-point. ◊

A general initial-condition. If we instead require that $\mathbf{y}(5)=\mathbf{P}$ for some particular pt $\mathbf{P}\in\mathbb{H}$, then just center ball $\mathbf{B}$ at $\mathbf{P}$ and replace (5) by

5′: $$\widehat{\mathbf{z}} \ :=\ \left[t\ \mapsto\ \mathbf{P}+\int_5^t f_{\mathbf{z}}(\tau)\,\mathrm{d}\tau\right].$$

Higher-order DEs

We first give an *example* of "re-coding" a higher-order DE to a 1st-order DE.

2nd-order to 1st-order. Over an interval $J\subset\mathbb{R}$, suppose we seek a soln $h: J\to\mathbb{R}$ to DE

8a: $$h''(t) \ =\ \sin\Big(t\cdot h'(t)\Big) + h(t)\,.$$

A fnc $\mathbf{x}: J\to\mathbb{R}^2$ can be written in components as

$$\mathbf{x}(t) \ =\ \begin{bmatrix} h_1(t)\\ h_0(t)\end{bmatrix}.$$

If each h_j is differentiable then so is $\mathbf{x}$, and $\mathbf{x}'$ equals $\begin{bmatrix} h_1\\ h_0\end{bmatrix}' \overset{\text{note}}{=} \begin{bmatrix} h_1'\\ h_0'\end{bmatrix}$. Consider DE (∗)

8b: $$\mathbf{x}'(t) \overset{\text{def}}{=} \begin{bmatrix} h_1(t)\\ h_0(t)\end{bmatrix}' \ \overset{*}{=}\ \begin{bmatrix} \sin(t\cdot h_1(t)) + h_0(t)\\ h_1(t)\end{bmatrix}.$$

The 0th-component, h_0, of a soln $\mathbf{x}$ to (8b) is automatically a soln to (8a). Conversely, a soln h to (8a) yields a soln $\begin{bmatrix} h_1()\\ h_0()\end{bmatrix}$ to (8b), by setting $h_0 := h$ and $h_1 := h'$.

Lastly, defining the kernel $\mathbf{K}: J\times\mathbb{R}^2\to\mathbb{R}^2$ by

8c: $$\mathbf{K}\left(t;\,\begin{bmatrix} h_1\\ h_0\end{bmatrix}\right) \ :=\ \begin{bmatrix} \sin(t\cdot h_1)+h_0\\ h_1\end{bmatrix},$$

makes the vector-valued DE $\mathbf{x}'(t) = \mathbf{K}\Big(t;\,\mathbf{x}(t)\Big)$ a restatement of (8b), hence of (8a).

Re-coding done more generally

Now consider a third-order DE OTForm

9: $$h''' = G(t;\, h, h', h'')\,, \quad \text{with } h^{(k)}(5) = P_k,\ \text{for } k = 0,1,2.$$

Here $h\colon J \to \mathbb{H}$ and $G\colon J \times \mathbb{H}^{\times 3} \to \mathbb{H}$ and each P_k is a point in $\mathbb{H}$. By letting h_k denote the k^{th} derivative $h^{(k)}$, we can restate the DE part of (9) as a vector equation

$$\begin{bmatrix} h_2' \\ h_1' \\ h_0' \end{bmatrix} = \begin{bmatrix} G(t;\, h_0, h_1, h_2) \\ h_2 \\ h_1 \end{bmatrix}.$$

Reduction. In order to write (9) as a *first*-order eqn, consider a fnc $\mathbf{x}\colon J \to \mathbb{H}^{\times 3}$ written as a column vector $\mathbf{x}(t) = \begin{bmatrix} h_2(t) \\ h_1(t) \\ h_0(t) \end{bmatrix}$, where each h_k maps $J \to \mathbb{H}$. This notation hands us a function

10: $$\mathbf{K}\colon J \times \mathbb{H}^{\times 3} \to \mathbb{H}^{\times 3} \quad \text{defined by} \quad \mathbf{K}\Big(t;\, \begin{bmatrix} h_2 \\ h_1 \\ h_0 \end{bmatrix}\Big) := \begin{bmatrix} G(t;\, h_0, h_1, h_2) \\ h_2 \\ h_1 \end{bmatrix}.$$

This allows us to rewrite (9) as

9′: $$\forall t \in J: \quad \mathbf{x}'(t) = \mathbf{K}\big(t;\, \mathbf{x}(t)\big)\,, \quad \text{with } \mathbf{x}(5) = \mathbf{P}\,,$$

where $\mathbf{P}$ denotes the point $\begin{bmatrix} P_2 \\ P_1 \\ P_0 \end{bmatrix}$ in $\mathbb{H}^{\times 3}$. ◊

Commentary. Suppose $\mathbf{K}$ is, say, 7-times continuously-differentiable. Then a fixed-pt of $\mathbf{z} \mapsto \widehat{\mathbf{z}}$, where

$$\widehat{\mathbf{z}}(t) := \mathbf{P} + \int_5^t \mathbf{K}\big(\tau;\, \mathbf{z}(\tau)\big)\, d\tau\,,$$

is (at least) 8-times continuously-diff'able.

When $\mathbf{K}()$ comes from (10) then the level of differentiability of $\mathbf{K}()$ is that of $G()$. □

When our hypotheses fail. Consider the following $\mathbb{H} := \mathbb{R}$ case. For C a real constant, define

11a: $$\mathbf{x}(t) = \mathbf{x}_C(t) := \frac{1}{1 + Ct}.$$

Note $\mathbf{x}(t) - 1 = \frac{-Ct}{1+Ct}$, so

$$\mathbf{x}(t) \cdot [\mathbf{x}(t) - 1] = \frac{-Ct}{[1 + Ct]^2}.$$

But this latter equals $t \cdot \mathbf{x}'(t)$. hence $\mathbf{x}_C()$ satisfies initial-value problem

11b: $$t \cdot \mathbf{x}'(t) = \mathbf{x}(t) \cdot [\mathbf{x}(t) - 1]\,, \quad \text{with} \quad \mathbf{x}(0) = 1\,.$$

For each $C \in [0,3)$ our $\mathbf{x}_C()$ is well-defined on interval $J := (-\frac{1}{3}, \infty)$. Since $J \ni 0$ [our initial-cond is at 0], we thus have an infinite family of solns to IVP (11b) on J; one soln for each $C \in [0,3)$.

Since the conclusion to FTODE is false, it must be that some FTODE-hypothesis failed.

Writing (11b) in form (1), our kernel is

11c: $$\mathbf{K}\big(t;\, x\big) := \frac{1}{t} \cdot [x^2 - x]\,.$$

But this is **not** well-defined at $t{=}0$, which is where our initial-condition takes place. Also, no matter how small a $t_0{>}0$ is take, the RhS(11c) is *not* $\mathbb{H}$-wise Lipschitz as t ranges over interval $(0, t_0)$. □