Algorithms for solving some differential equations [v.8]

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✓Quiz S: *Wedn., 03Feb. 2021*
✓Quiz T: *Wedn., 17Feb.*
✓Quiz U: *Wedn., 03Mar.*
✓Quiz V: *Wedn., 17Mar.*
✓Quiz W: *Wedn., 31Mar.*
Quiz X: *Tuesday, 20Apr.*

What does this mean?

stand	*took*	*to*	*world.*
I	*you*	*throw*	*the*

Number Sets. Expression $k \in \mathbb{N}$ [read as "k is an element of $\mathbb{N}$" or "k in $\mathbb{N}$"] means that k is a natural number; a ***natnum***. Expression $\mathbb{N} \ni k$ [read as "$\mathbb{N}$ ***owns*** k"] is a synonym for $k \in \mathbb{N}$.

$\mathbb{N}$ = natural numbers = $\{0, 1, 2, \dots\}$.

$\mathbb{Z}$ = integers = $\{\dots, -2, -1, 0, 1, \dots\}$. For the set $\{1, 2, 3, \dots\}$ of positive integers, the ***posints***, use $\mathbb{Z}_+$. Use $\mathbb{Z}_-$ for the negative integers, the ***negints***.

$\mathbb{Q}$ = rational numbers = $\{\frac{p}{q} \mid p \in \mathbb{Z} \text{ and } q \in \mathbb{Z}_+\}$. Use $\mathbb{Q}_+$ for the positive rationals and $\mathbb{Q}_-$ for the negative rationals.

$\mathbb{R}$ = reals. The ***posreals*** $\mathbb{R}_+$ and the ***negreals*** $\mathbb{R}_-$.

$\mathbb{C}$ = complex numbers, also called the ***complexes***.

For $\omega \in \mathbb{C}$, let "$\omega > 5$" mean "ω *is* **real** and $\omega > 5$". [Use the same convention for $\geq, <, \leq$, and also if 5 is replaced by any real number.]

Use $\overline{\mathbb{R}} = [-\infty, +\infty] := \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$, the ***extended reals***.

An "***interval of integers***" $[b\,..\,c)$ means the intersection $[b, c) \cap \mathbb{Z}$; ditto for open and closed intervals. So $[\mathrm{e}\,..\,2\pi] = \{3, 4, 5, 6\} = [3\,..\,6] = (2\,..\,6]$. We allow b and c to be $\pm\infty$; so $(-\infty\,..\,-1]$ is $\mathbb{Z}_-$. And $[-\infty\,..\,-1]$, is $\{-\infty\} \cup \mathbb{Z}_-$.

Floor function: $\lfloor \pi \rfloor = 3$, $\lfloor -\pi \rfloor = -4$. Ceiling fnc: $\lceil \pi \rceil = 4$. Absolute value: $|-6| = 6 = |6|$ and $|-5 + 2\mathrm{i}| = \sqrt{29}$.

Mathematical objects. HI Seq: '*sequence*'. poly(s): '*polynomial(s)*'. irred: '*irreducible*'. Coeff: '*coefficient*' and var(s): '*variable(s)*' and parm(s): '*parameter(s)*'. Expr.: '*expression*'. Fnc: '*function*' (so ratfnc: means rational function, a ratio of polynomials). trnfn: '*transformation*'. cty: '*continuity*'. cts: '*continuous*'. diff'able: '*differentiable*'. CoV: '*Change-of-Variable*'. CoI: '*Constant of Integration*'. LoI: '*Limit(s) of Integration*'. RoC: '*Radius of Convergence*'.

Soln: '*Solution*'. Thm: '*Theorem*'. Prop'n: '*Proposition*'. CEX: '*Counterexample*'. eqn: '*equation*'. RhS: '*RightHand side*' of an eqn or inequality. LhS: '*lefthand side*'. Sqrt or Sqroot: '*square-root*', e.g, "the sqroot of 16 is 4". Ptn: '*partition*', *but* pt: '*point*' as in "a fixed-pt of a map".

FTC: '*Fund. Thm of Calculus*'. IVT: '*intermediate-Value Thm*'. MVT: '*Mean-Value Thm*'.

The ***logarithm*** function, defined for $x>0$, is $\log(x) := \int_1^x \frac{\mathrm{d}v}{v}$. Its inverse-fnc is $\exp()$.

For $x>0$, then, $\exp(\log(x)) = x = \mathrm{e}^{\log(x)}$. For real t, naturally, $\log(\exp(t)) = t = \log(\mathrm{e}^t)$.

PolyExp: '*Polynomial-times-exponential*', e.g, $[3 + t^2]\cdot\mathrm{e}^{4t}$. PolyExp-sum: '*Sum of polyexps*'. E.g, $f(t) := 3t\mathrm{e}^{2t} + [t^2]\cdot\mathrm{e}^t$ is a polyexp-sum.

Prefix nt- means '*non-trivial*'. E.g "a *nt*-soln to $f' = 5f$ is $f(t) := \mathrm{e}^{5t}$; a *trivial* soln is $f \equiv 0$."

Phrases. WLOG: '*Without loss of generality*'. IFF: '*if and only if*'. TFAE: '*The following are equivalent*'. ITOf: '*In Terms Of*'. OTForm: '*of the form*'. FTSOC: '*For the sake of contradiction*'. And ※="Contradiction".

IST: '*It Suffices To*', as in ISTShow, ISTExhibit.

Use w.r.t: '*with respect to*' and s.t: '*such that*'.

Latin: e.g: *exempli gratia*, '*for example*'. i.e: *id est*, '*that is*'. N.B: *Nota bene*, '*Note well*'. inter alia: '*among other things*'. QED: *quod erat demonstrandum*, meaning "end of proof".

Factorial. Def: $n! := n\cdot[n-1]\cdot[n-2]\cdots 2\cdot 1$; so $0! = 1$.

Rising Fctrl: $[\![x \uparrow K]\!] := x\cdot[x+1]\cdot[x+2]\cdots[x+[K-1]]$,

Falling Fctrl: $[\![x \downarrow K]\!] := x\cdot[x-1]\cdot[x-2]\cdots[x-[K-1]]$,

for natnum K and $x \in \mathbb{C}$. E.g, $[\![K \downarrow K]\!] = K! = [\![1 \uparrow K]\!]$.

N.B: For $n \in \mathbb{N}$: If $K > n$ then $[\![n \downarrow K]\!] = 0$.

Note $[\![x \uparrow K]\!] = [\![x + [K-1] \downarrow K]\!]$.

Learn from the mistakes of others. You can't live long enough to make them all yourself.
—Eleanor Roosevelt

Some differentiation formulas. Below, italic boldface parameters $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ and $\boldsymbol{f}$ represent *numbers*. Here, differentiation is w.r.t variable t.

1.1: $$t \cdot \mathrm{e}^{t/\boldsymbol{c}} = \left[\mathrm{e}^{t/\boldsymbol{c}} \cdot [\boldsymbol{c}t - \boldsymbol{c}^2]\right]'.$$

1.2: $$t^2 \cdot \mathrm{e}^{t/\boldsymbol{c}} = \left[\mathrm{e}^{t/\boldsymbol{c}} \cdot [\boldsymbol{c}t^2 - 2\boldsymbol{c}^2 t + 2\boldsymbol{c}^3]\right]'.$$

1.3: $$\frac{\boldsymbol{c}}{\boldsymbol{a} + \boldsymbol{b}t} = \left[\tfrac{\boldsymbol{c}}{\boldsymbol{b}} \cdot \log(\boldsymbol{a} + \boldsymbol{b}t)\right]'.$$

Use expressions $E(t) := \mathrm{e}^{\boldsymbol{a}t}$, $S(t) := \sin(\boldsymbol{f}\cdot t)$ and $C(t) := \cos(\boldsymbol{f}\cdot t)$, below. The number $\boldsymbol{f}$ can be thought of as "frequency" and, in some contexts, the $\boldsymbol{a}$ can be thought of as "attenuation". We have

1.4: $$[\boldsymbol{a}^2 + \boldsymbol{f}^2] \cdot \int E\cdot S = E \cdot [\boldsymbol{a}S - \boldsymbol{f}C].$$

1.5: $$[\boldsymbol{a}^2 + \boldsymbol{f}^2] \cdot \int E\cdot C = E \cdot [\boldsymbol{f}S + \boldsymbol{a}C].$$

§Table of Contents (ToC)

Introduction

[Use **NSS9** for the 9th edition of the *Nagle,Saff,Snider* textbook. Use, e.g, **#7**^{P}*193*.**NSS9**, to refer to problem #7 on page 193 of **NSS9**.] [Use **ZW 8** for 8th edition of *Zill* & *Wright*, using e.g, **#7**^{P}*193*.ZW8, to refer to problems.]

For the following algorithms, the *unknown function* is $y = y(t)$. For a DE of form

$$\mathrm{Fnc}(y, y', y'', \ldots) \quad = \quad G(),$$

we will call $G()$ the ***tarGet fnc***.

Use $\mathbf{D}$ for the ***differentiation operator***; therefore $\mathbf{I} := \mathbf{D}^0$ is the ***identity operator***. And $\mathbf{D}^3(y)$ means $\mathbf{D}\Big(\mathbf{D}(\mathbf{D}(y))\Big)$, i.e y'''. So $\mathbf{I}(y) = \mathbf{D}^0(y) = y$. [See §5.2–**NSS9**, P.243.]

Use DE: '*Differential Equation*', LDE: '*Linear DE*', ODE: '*Ordinary DE*' and PDE: '*Partial DE*'. IVP: '*Initial-Value Problem*'.

Use boldface $\mathbf{1} := [t \mapsto 1]$, for the constant-1 fnc. For the ***identity function***, use $Id(t) = t$. Differentiating, $Id' = \mathbf{1}$.

The Easy Scan

Below, $\alpha, \beta, A, B, \mathbf{r}$ range over all numbers; $\mathbb{R}$ or $\mathbb{C}$, as appropriate.

Before we work on solving a DE with U.F $y(t)$, let's glean some properties of $\mathcal{S}$, the soln-set of the DE.

What is the name of: *The indep.var? The U.F? What are the parameters in the DE?* And: *What is the order of the DE?*

Types of functions.

a1**:** *Is the zero-fnc a soln? Are there constant-solns?*

a2**:** *Are there non-constant polynomial solns?* [This usually involves examining how the DiffOp affects the degree of a polynomial.]

a3**:** *Could a nt-exponential,* $A{\cdot}\mathrm{e}^{Bt}$ *with* $B{\neq}0$ *and* $A{\neq}0$*, be a soln to the DE?*

Closure properties of $\mathcal{S}$.

b1: *Is $\mathcal{S}$ sealed [closed] under horizontal translation? I.e, for soln f and number $\mathbf{r}$, must $\mathbf{T}_{\mathbf{r}}(f)$ also be a soln? I.e, is the DE autonomous?*

b2: *Is $\mathcal{S}$ sealed [closed] under scaling?, i.e, for $f \in \mathcal{S}$, must each αf also be a soln?*

For $f,g \in \mathcal{S}$, must $f+g \in \mathcal{S}$?

[This $\mathcal{S}$ is sealed under scaling *and* under addition IFF the DE can be written in form $\text{LinearOp}(y) = 0$.]

b3: *If not* (b2), *then is $\mathcal{S}$ at least sealed under* ***averaging****? I.e, $\forall f,g \in \mathcal{S}$ and all scalars α,β with $\alpha + \beta = 1$, is average $\left[[\alpha f] + \beta g\right]$ a soln?*

b4: *Special? Is the DOp linear, affine, equidimensional, a CCLDOp? Is the DE autonomous, separable, EXACT*(ifiable)*, FOLDE, Bernoulli-type?*

Easy-Scan Example. Consider U.F $y=y(t)$ satisfying

$$*: \quad \frac{dy}{dt} = 6t^2 \cdot [y-4].$$

Checking types (a1,a2,a3). For analysis, define operators [Left and Right]

$$\mathsf{L}(y) := \frac{dy}{dt}; \qquad [\text{so here, } \mathsf{L} = \mathbf{D}]$$
$$\mathsf{R}(y) := 6t^2 \cdot [y-4].$$

Since $\mathsf{L}(y)\equiv 0$ IFF $y\equiv$*Constant*, *the only constant soln to* $(*)$ *is* $y\equiv 4$. And for a poly y of degree $N\geq 1$, necessarily $\text{Deg}(\mathsf{R}(y)) = 2+N$, whereas $\text{Deg}(\mathsf{L}(y)) = N-1$. *So* <u>*no*</u> *non-constant polynomial solns.*

Lastly, $\mathsf{L}(A\cdot e^{Bt})$ equals [with $A,B \neq 0$] another nt-exponential, $AB\cdot e^{Bt}$. But $\mathsf{R}(A\cdot e^{Bt})$ is not a pure exponential, because of the polynomial factor. *So* $(*)$ *has* <u>*no*</u> *nt-exponential solns.*

Checking closure properties. More to come... □

Soln to $(*)$. Our DE is separable, so we can get at least an *implicit* $(*)$-soln. Because we did the Easy-scan *first*, should our computation yield a non-trivial polynomial or exponential answer, then we erred *either* in our SoV computation... *or* in our Easy-scan... *OR* both!

Separating $(*)$ gives $\frac{1}{y-4}\,dy = 6t^2\,dt$. Let's only consider real solns $y()$ with $y()>4$. [I'm avoiding discussing what it means to extend log() to $\mathbb{C}$.] Using CoI α, antidiffing yields

$$\log(y-4) = \alpha + 2t^3. \quad \textit{Exponentiating,}$$
$$y = 4 + [e^\alpha \cdot e^{2t^3}].$$

Renaming $\beta := e^\alpha$, then, gives

$$**: \quad y_\beta(t) = 4 + \beta e^{2t^3}.$$

Indeed, *each* $\beta\in\mathbb{C}$ has $(**)$ satisfy $(*)$. Let's <u>*check*</u>...

Does $(**)$ satisfy $(*)$? Abbreviating $E := e^{2t^3}$, note $E' \overset{\text{Chain rule}}{=\!=} E\cdot 2{\cdot}3t^2$, i.e $E' = 6t^2E$. Thus

$$[\text{RhS}(*)]' = y' = \beta\cdot 6t^2E.$$

Note $y-4 = \beta E$. So $\text{LhS}(*) \overset{\text{def}}{=} 6t^2\cdot\beta E$. This indeed equals $[\text{RhS}(*)]'$, as desired. □

I am always ready to learn although I do not always like being taught.
–Winston Churchill

There's a delta for every epsilon

It's a fact that you can always count upon.
There's a delta for every epsilon
And now and again,
There's also an N.

But one condition I must give:
The epsilon must be positive
A lonely life all the others live,
In no theorem
A delta for them.

How sad, how cruel, how tragic,
How pitiful, and other adjec-
Tives that I might mention.
The matter merits our attention.
If an epsilon is a hero,
Just because it is greater than zero,
It must be mighty discouragin'
To lie to the left of the origin.

This rank discrimination is not for us,
We must fight for an enlightened calculus,
Where epsilons all, both minus and plus,
Have deltas
To call their own.

Words and Music by: *–Tom Lehrer*

Video of Lehrer performing the δ-ε song.

Lyrics, and audio of Lehrer performing.

Separation of variables [SoV]

Consider UF $f{=}f(t)$ defined on interval $\mathbb{J} := [-3, 7]$ satisfying IVP

2a: $$f'(t) = \boldsymbol{\beta}(t)/\boldsymbol{\mu}(f(t)),$$ with $f(5) = 9$.

Let $\mathbb{K} := f(\mathbb{J})$, the interval which is the f-image of $\mathbb{J}$.

Together with functions $\boldsymbol{\beta}, \boldsymbol{\mu}$, suppose we have three other fncs $\mathsf{B}, \mathsf{M}, \mathsf{M}^{\text{InvF}}$ satisfying:

2b:
Fncs $\boldsymbol{\beta}, \mathsf{B}$ are defined on $\mathbb{J}$, with $\mathsf{B}' = \boldsymbol{\beta}$.
Fncs $\boldsymbol{\mu}, \mathsf{M}$ are defined on $\mathbb{K}$, with $\boldsymbol{\mu}{\neq}0$ and $\mathsf{M}' = \boldsymbol{\mu}$.
Fnc M is invertible with M^{InvF} its inverse-fnc.

Re-write the top-line of (2a) as

2c: $$\boldsymbol{\mu}(f(t)) \cdot f'(t) = \boldsymbol{\beta}(t).$$

For each $t \in \mathbb{J}$, then, we have

2d: $$\int_5^t \boldsymbol{\mu}(f(t)) \cdot f'(t)\,\mathrm{d}t = \int_5^t \boldsymbol{\beta}(t)\,\mathrm{d}t.$$

Substitution $y = f(t)$ says LhS(2d) equals

$$\int_{f(5)}^{f(t)} \boldsymbol{\mu}(y) \cdot \mathrm{d}y \overset{\text{by FTC}}{=\!=} \mathsf{M}(f(t)) - \mathsf{M}(f(5)).$$

And RhS(2d) equals $\mathsf{B}(t) - \mathsf{B}(5)$. Hence

$$\mathsf{M}(f(t)) = \mathsf{B}(t) + [\mathsf{M}(f(5)) - \mathsf{B}(5)].$$

Consequently, initial condition (2a) produces

2e: $$f(t) = \mathsf{M}^{\text{InvF}}\Big(\mathsf{B}(t) + \mathsf{M}(9) - \mathsf{B}(5)\Big).$$

Example of SoV. Consider U.F. $f{=}f(t)$ satisfying

2a†: $$f'(t) = \mathsf{e}^{-2f(t)} \cdot t \overset{\text{note}}{=\!=} 2t/2\mathsf{e}^{2f(t)},$$ with $f(0) = 9$.

Soln. [Do Easy-Scan first.] Define the following fncs:

2b†:
$$\boldsymbol{\beta}(t) := 2t \quad\text{and}\quad \mathsf{B}(t) := t^2.$$
$$\boldsymbol{\mu}(y) := 2\mathsf{e}^{2y} \quad\text{and}\quad \mathsf{M}(y) := \mathsf{e}^{2y}.$$
$$\text{Hence}\quad \mathsf{M}^{\text{InvF}} = \tfrac{1}{2} \cdot \log.$$

Computing, $\mathsf{B}(t) + \mathsf{M}(9) - \mathsf{B}(0) = t^2 + \mathsf{e}^{18} - 0$. Hence

2e†: $$f(t) = \tfrac{1}{2}\log(t^2 + \mathsf{e}^{18}).$$

Check. To verify that (2e†) satisfies (2a†), note

$*$: $$f'(t) = \frac{2t + 0}{2 \cdot [t^2 + \mathsf{e}^{18}]} \overset{\text{note}}{=\!=} \frac{t}{t^2 + \mathsf{e}^{18}}.$$

And $\mathsf{e}^{2f(t)} = \mathsf{e}^{\log(t^2 + \mathsf{e}^{18})} = t^2 + \mathsf{e}^{18}$. Hence $\mathsf{e}^{-2f(t)} \cdot t$ equals $t/[t^2 + \mathsf{e}^{18}]$, which indeed equals RhS($*$).

Finally, to verify the initial condition, note $f(0)$ equals $\frac{1}{2}\log(\mathsf{e}^{18}) = \frac{1}{2} \cdot 18 = 9$.

CoV to SoV

A function $F(x_1, \ldots, x_N)$ is ***scale-invariant*** [or "***homogeneous*** of degree-0"] if

3.1: $\forall s \neq 0 : \; F(sx_1, \ldots, sx_N) \; = \; F(x_1, \ldots, x_N)$.

[I.e, F is unchanged by scaling.] More generally, for a $\mathbf{d} \in \mathbb{R}$, say that $F()$ is "***homogeneous*** of degree **d**" if

3.2: $\forall s \neq 0 : \; F(sx_1, \ldots, sx_N) \; = \; s^{\mathbf{d}} \cdot F(x_1, \ldots, x_N)$.

3.3: Scale-invariant to SoV. Consider a scale-invariant $F(x, y)$, U.F $y = y(x)$, and DE

3.3a: $\frac{dy}{dx} \; = \; F(x, y)$.

Define CoV $\boxed{v := \frac{y}{x}}$ and fnc $G(v) := F(1, v)$. Solve

3.3b: $$\frac{1}{G(v) - v} \cdot dv \; = \; \frac{1}{x} \cdot dx$$

using SoV. For each number α, then,

3.3c: $$y_\alpha(x) \; := \; x \cdot v_\alpha(x)$$

solves (3.3a). [**NB:** You might only obtain *implicit* solns.] □

Why does this work? Substitution $v := \frac{y}{x}$ yields that
$$F(x, y) \; = \; F(1, \tfrac{y}{x}) \stackrel{\text{note}}{=\!=} G(v).$$

Rewrite $v := \frac{y}{x}$ as $y = x \cdot v$. The Product Rule gives
$$G(v) \; = \; \frac{dy}{dx} \stackrel{\text{P.R.}}{=\!=} 1 \cdot v \; + \; x \cdot \frac{dv}{dx}.$$

This separable DE, rewritten, is (3.3b).

Scale-invariant CoV Example. To compute U.F $y = y(x)$, divide by x in DE
$$x \cdot \frac{dy}{dx} \; = \; x + 5y, \qquad \textit{obtaining}$$

3.3a†: $$\frac{dy}{dx} \; = \; 1 + 5 \cdot \frac{y}{x}. \quad \begin{bmatrix}\text{Note RhS is} \\ \text{scale-invariant.}\end{bmatrix}$$

So define $G(v) := 1 + 5v$. Then $G(v) - v \; = \; 1 + 4v$. So (3.3b) becomes

3.3b†: $$\frac{1}{1 + 4v} \cdot dv \; = \; \frac{1}{x} \cdot dx.$$

Integrating each side, using α as CoI, produces
$$\tfrac{1}{4} \log(|1 + 4v|) \; = \; \alpha + \log(|x|).$$

Letting $\beta := 4\alpha$ gives
$$\log(|1 + 4v|) \; = \; \beta + 4\log(|x|).$$

Exponentiating,
$$|1 + 4v| \; = \; e^\beta \cdot |x|^4.$$

With $\gamma := \pm e^\beta$, discard the abs.values, obtaining
$$1 + 4v \; = \; \gamma \cdot x^4.$$

Recovering y, we now have that
$$\frac{y}{x} \stackrel{\text{def}}{=\!=} v \; = \; \tfrac{1}{4}\gamma x^4 - \tfrac{1}{4}.$$

With $\sigma := \frac{1}{4}\gamma$, multiplying both sides by x delivers

3.3c†: $$y_\sigma(x) \; = \; \sigma x^5 - \tfrac{1}{4} x.$$

Checking. Does (3.3c†) satisfy $x \cdot \frac{dy}{dx} = x + 5y$? Computing its LhS,

$*$: $$x \cdot \frac{dy}{dx} \; = \; x \cdot [5\sigma x^4 - \tfrac{1}{4}] \; = \; 5\sigma x^5 - \tfrac{1}{4} x.$$

Again using (3.3c†),
$$x + 5y \; = \; x + [5\sigma x^5 - \tfrac{5}{4} x] \stackrel{\text{note}}{=\!=} \text{RhS}(*). \qquad \square$$

Scale-invar. CoV Ex.2. For $t>0$, U.F $y=y(t)$ satisfies

$$*: \quad y^2 y' t \;=\; t^3 + y^3. \quad \textit{Dividing by } y^2 t \textit{ produces}$$

$$3.3a‡: \quad y' \;=\; \left[\frac{t}{y}\right]^2 + \frac{y}{t}. \qquad \text{Note RhS is scale-invariant.}$$

With $v := \frac{y}{t}$, then, this RhS is $G(v) := \frac{1}{v^2} + v$. Then $G(v) - v = \frac{1}{v^2}$. So (3.3b) becomes

$$3.3b‡: \quad v^2 \cdot \mathrm{d}v \;=\; \frac{1}{t} \cdot \mathrm{d}t.$$

Integrating each side, using α as CoI, produces

$$\tfrac{1}{3}v^3 \;=\; \alpha + \log(t).$$

Let $\beta := 3\alpha$. Then

$$v \;=\; \Big[\beta + 3\log(t)\Big]^{1/3}.$$

Consequently,

$$3.3c‡: \quad y_\beta(t) \;=\; t \cdot \Big[\beta + 3\log(t)\Big]^{1/3}.$$

Checking. Does (3.3c‡) satisfy $(*)$?
With $\mathrm{S} := [\beta + 3\log(t)]$, note

$$**: \quad y' \;=\; [t \cdot \mathrm{S}^{1/3}]' \;=\; 1 \cdot \mathrm{S}^{1/3} + t \cdot \tfrac{1}{3}\mathrm{S}^{-2/3} \cdot \tfrac{3}{t} \;=\; \mathrm{S}^{1/3} + \mathrm{S}^{-2/3}.$$

Multiplying $(**)$ by $y^2 t \overset{\text{note}}{=\!=} t^3\,\mathrm{S}^{2/3}$ yields

$$y^2 y' t \;=\; \big[t\,\mathrm{S}^{1/3}\big]^3 + t^3 \overset{\text{note}}{=\!=} y^3 + t^3. \quad \checkmark \quad \square$$

3.4: Linear-CoV to SoV. A function $H()$, and numbers P,Q, define DE

3.4a: $$\frac{dy}{dx} = H(Px + Qy).$$

WLOG, $Q \neq 0$. CoV $\boxed{z := Px + Qy}$ implies that

$$\frac{dz}{dx} = P \cdot 1 + Q \cdot \frac{dy}{dx} \stackrel{\text{note}}{=\!=} P + Q \cdot H(z).$$

Apply SoV to

3.4b: $$\frac{1}{P + Q \cdot H(z)} \cdot dz = 1 \cdot dx.$$

Each number α, then, gives a soln

3.4c: $$y_\alpha(x) := [z_\alpha(x) - P \cdot x]/Q$$

to (3.4a). [These solns might only be implicit solns.] □

Linear-CoV to SoV Example. Consider U.F $y = y(t)$ fulfilling

3.4a†: $$\frac{dy}{dt} = \exp(t + y).$$

Setting $z := t + y$, note $\frac{dz}{dt} = 1 + \frac{dy}{dt} = 1 + e^z$. Hence $\frac{dz}{1+e^z} = 1 \cdot dt$. Anti-diffing gives

$$z - \log(1 + e^z) = \alpha + t,$$

for CoI α. While we do not know how to solve this implicit soln *explicitly*, we can rewrite it for y as

3.4c†: $$y + t - \log\Big(1 + \exp(y + t)\Big) = \alpha + t.$$

By applying $\frac{d}{dt}$, the energetic reader can verify that this *is* an implicit soln to (3.4a†).

Complex numbers

[Complex arithmetic done in class.]

> *The number you have reached is imaginary. Please rotate your phone 90 degrees and dial again.*
>
> *–David Grabiner*

4.1: SV Buried Treasure Problem [BTP]. *Floating in the ocean you spy a bottle containing a pirate's map to fabulous treasure. You sell your possessions, purchase a robot-crewed ocean-catamaran, and sail to the island, discovering it is a vast plateau. The map says:*

Arrrgh, Matey! Count your paces from the gallows to the a quartz boulder, turn Left 90° and walk the same distance; hammer a gold spike into the ground.

Count your paces from the gallows to the giant oak, turn Right 90° and walk the counted distance; hammer a silver spike into the ground.

Find Ye Buried Treasure midway between the spikes.

With joy, you bound up the plateau [with the treasure you can say *bye bye* to *annoying Math classes!*] *and immediately spot the giant oak, and quartz boulder. But the gallows has rotted away without a trace.*

Nonetheless, you find the Treasure. How? ◊

[Hint: Using B, K, w for the Bolder's, oaK's and (unknown) galloWs' location, write the treasure's spot as a fnc $\mathbf{t}_{B,K}(w)$ by using $\mathbb{C}$ addition and multiplication.] Alphabetic-order mnemonic:

Boulder	Left	gold
oaK	Right	silver

Solved by: *Matthew C, Junhao Z., Hani S.*, 2020t. *Nathan T.*, 2021t.

(*Partial soln*) *Sreeram V.*, 2022g. *Maxime A.*, 2023g.

Remark. The ***discriminant*** of quadratic [i.e, $A \neq 0$] polynomial $q(z) := Az^2 + Bz + C$ is

5.1: $$\text{Discr}(q) := B^2 - 4AC.$$

The zeros ["roots"] of q are

5.2: $$\text{Roots}(q) = \frac{1}{2A}\Big[-B \pm \sqrt{\text{Discr}(q)}\,\Big].$$

Hence when A,B,C are *real*, then the zeros of q form a complex-conjugate pair. And q has a *repeated root* IFF $\text{Discr}(q)$ is zero.

A monic $\mathbb{R}$-irreducible quadratic has form

5.3: $$q(z) = z^2 - \mathcal{S}z + \mathcal{P} = [z - \mathbf{r}] \cdot [z - \overline{\mathbf{r}}],$$

where $\mathbf{r} \in \mathbb{C}\setminus\mathbb{R}$. Note $\mathcal{S} = \mathbf{r} + \overline{\mathbf{r}} = 2\mathrm{Re}(\mathbf{r})$ is the *Sum* of the roots. And $\mathcal{P} = \mathbf{r}\cdot\overline{\mathbf{r}} = |\mathbf{r}|^2$ is the *Product* of the roots. The g discriminant, $\mathrm{Discr}(g)$, equals

5.4: $$\mathcal{S}^2 - 4\mathcal{P} \overset{\text{note}}{=\!=} [\mathbf{r} - \overline{\mathbf{r}}]^2 = -4\cdot[\mathrm{Im}(\mathbf{r})]^2 .$$

Completing-the-square yields

5.5: $$q(z) = \left[z - \tfrac{\mathcal{S}}{2}\right]^2 + F^2, \text{ where } F := |\mathrm{Im}(\mathbf{r})|,$$

which is easily checked. [Exercise] □

6: Fundamental Theorem of Algebra (Gauss and friends). *Consider a monic ℂ-polynomial*

$$p(z) := z^N + B_{N-1}z^{N-1} + \ldots + B_1 z + B_0 .$$

Then p factors completely over ℂ as

$$p(z) = [z - \mathbf{r}_1]\cdot[z - \mathbf{r}_2]\cdot\ldots\cdot[z - \mathbf{r}_N],$$

for a list $\mathbf{r}_1,\ldots,\mathbf{r}_N \in \mathbb{C}$, possibly with repetitions. This list is unique up to reordering.

*If p is a **real** polynomial, i.e $\overline{p} = p$, then p factors over ℝ as a product of monic ℝ-irreducible linear and ℝ-irred. quadratic polynomials. The product is unique up to reordering.*

Also: A proof-sketch is in Primer on Polynomials on my Teaching page. ◊

ℂ-exponential [Chap4–NSS9, P.237]

The algebraic structure of ℝ can be consistently extended to a larger field, by adjoining a sqroot of negative 1. This is conventionally[♡1] called $\mathbf{i}$, so $\mathbf{i}^2 = -1 = [-\mathbf{i}]^2$. Extending ℝ by $\mathbf{i}$ produces field

$$\mathbb{C} := \{x\mathbf{1} + y\mathbf{i} \mid \textit{where } x \textit{ and } y \textit{ are real}\}.$$

[I've written $x\mathbf{1} + y\mathbf{i}$ to emphasize that the additive structure of ℂ is that of a 2-dimensional ℝ-vectorspace, with basis vectors $\mathbf{1}$ and $\mathbf{i}$. In practice, we write $2+3\mathbf{i}$, not $2\cdot\mathbf{1} + 3\mathbf{i}$.]

A geometric picture of ℂ, with the ***real axis*** horizontal, and the ***imaginary axis*** vertical, is called the ***Argand plane*** or the ***complex plane***.

[♡1] Electrical engineers use $\mathbf{j}$ rather than $\mathbf{i}$, as "i" is used to represent current/amperage in EE. Also, while boldface $\mathbf{i}$ is <u>a</u> sqroot of −1, we still have non-boldface i as a variable. E.g, we *could* [but wouldn't] write $7\mathbf{i} + \sum_{i=3}^{4} i^2 \overset{\text{note}}{=\!=} 7\mathbf{i} + 3^2 + 4^2$.

Write ***real-part*** and ***imaginary-part*** extractors as, e.g, for $z := 2 - 3\mathbf{i}$, give

$$\mathrm{Re}(z) = 2 \quad \textit{and} \quad \mathrm{Im}(z) = -3$$

since $z = 2\cdot 1 + [-3]\cdot\mathbf{i}$. The ***absolute-value*** or ***modulus*** of z is its distance to the origin; so

$$|z| = \sqrt{\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2}\,.$$

[Here, $|2 - 3\mathbf{i}| = \sqrt{4+9} = \sqrt{13}$.] The ***complex conjugate*** of this z is $\overline{z} = 2 + 3\mathbf{i}$. For a general $\omega = x + y\mathbf{i}$ with $x,y \in \mathbb{R}$, observe that

$$\mathrm{Re}(\omega) := x = \tfrac{\omega+\overline{\omega}}{2}, \quad \mathrm{Im}(\omega) := y = \tfrac{\omega-\overline{\omega}}{2\mathbf{i}};$$
$$\overline{\omega} = \mathrm{Re}(\omega) - \mathrm{Im}(\omega)\mathbf{i};$$
$$|\omega|^2 \overset{\text{Pythag. thm}}{=\!=\!=} x^2 + y^2 = \omega\overline{\omega}.$$

(Complex-)conjugation $\omega \mapsto \overline{\omega}$ is an *involution* of ℂ, since $\overline{\overline{\omega}} = \omega$. For complex polynomial $f(z) = \sum_{j=0}^{N} \mathbf{c}_j z^j$, define $\overline{f}(z) := \sum_{j=0}^{N} \overline{\mathbf{c}_j}\, z^j$, its ***conjugate polynomial***. Thus

$$\overline{f(z)} = \overline{f}(\overline{z}),$$

since $\overline{\mu+\nu} = \overline{\mu} + \overline{\nu}$ and $\overline{\mu\nu} = \overline{\mu}\cdot\overline{\nu}$ for $\mu,\nu \in \mathbb{C}$.

Multiplying complex numbers corresponds to multiplying their moduli and adding their angles.

To write a quotient $\frac{\nu}{\alpha}$ in std $x + \mathbf{i}y$ form, note

$$\frac{\nu}{\alpha} = \frac{\nu\overline{\alpha}}{\alpha\overline{\alpha}} = \nu\overline{\alpha}/|\alpha|^2$$

So write $\nu\overline{\alpha}$ in std form, then divide by real $|\alpha|^2$.

See *W: Complex number* and *W: Argand plane* for arithmetic with complex numbers.

Let's extend the exponential fnc to ℂ.

7a: Defn. *For $z \in \mathbb{C}$, define*

$$\exp(z) := \mathrm{e}^z := \sum_{n=0}^{\infty} \frac{1}{n!}\cdot z^n = 1 + z + \tfrac{1}{2}z^2 + \tfrac{1}{6}z^3 + \ldots;$$
$$\cos(z) := \sum_{k=0}^{\infty} \frac{[-1]^k}{[2k]!}\cdot z^{2k} = 1 - \tfrac{1}{2}z^2 + \tfrac{1}{24}z^4 - \ldots;$$
$$\sin(z) := \sum_{k=0}^{\infty} \frac{[-1]^k}{[2k+1]!}\cdot z^{2k+1} = z - \tfrac{1}{6}z^3 + \tfrac{1}{120}z^5 - \ldots.$$

Each series has ∞-RoC. ◊

Since we have absolute convergence of each series, we can re-order terms without changing convergence.

7b: Lemma. *Fix $\alpha,\beta \in \mathbb{C}$. Then*

$$e^{\alpha} \cdot e^{\beta} = e^{\alpha+\beta}. \qquad \lozenge$$

Proof. For natnum N, recall the Binomial thm which says that

$$*: \quad \sum_{j+k=N} \binom{N}{j,k} \cdot \alpha^j \beta^k = [\alpha+\beta]^N,$$

where the sum is over all ordered-pairs (j,k) of natnums. By its defn [and abs.convergence], $e^{\alpha}e^{\beta}$ equals

$$\Bigl[\sum_{j=0}^{\infty} \frac{1}{j!} \cdot \alpha^j\Bigr] \cdot \Bigl[\sum_{k=0}^{\infty} \frac{1}{k!} \cdot \beta^k\Bigr] = \sum_{N=0}^{\infty} \Bigl[\sum_{j+k=N} \frac{1}{j!}\frac{1}{k!} \cdot \alpha^j \beta^k\Bigr].$$

But $\frac{1}{j!\cdot k!}$ equals $\frac{1}{N!} \cdot \frac{N!}{j!\cdot k!}$. Hence $e^{\alpha}e^{\beta}$ equals

$$\sum_{N=0}^{\infty} \frac{1}{N!}\Bigl[\sum_{j+k=N} \binom{N}{j,k} \cdot \alpha^j \beta^k\Bigr] \overset{\text{by } (*)}{=\!=} \sum_{N=0}^{\infty} \frac{1}{N!}[\alpha+\beta]^N,$$

which is the defn of $e^{\alpha+\beta}$. ♦

7c: Lemma. *For θ,x,y,z complex numbers:*

7.1: $e^{i\theta} = [\cos(\theta) + i\sin(\theta)] =: \operatorname{cis}(\theta)$. *Hence*

7.2: $\dfrac{e^{i\theta} + e^{-i\theta}}{2} = \cos(\theta)$, $\dfrac{e^{i\theta} - e^{-i\theta}}{2i} = \sin(\theta)$. *Also,*

7.3: $e^{x \pm iy} = e^{x} \cdot e^{\pm iy} = e^{x} \cdot [\cos(y) \pm i\sin(y)]$,

since $\cos(-y) = \cos(y)$ and $\sin(-y) = -\sin(y)$.
When θ is real, then,

7.4: $\operatorname{Re}(e^{i\theta}) = \cos(\theta)$ *and* $\operatorname{Im}(e^{i\theta}) = \sin(\theta)$.

Since the coefficients in their power-series expansions are all real, our exp(),cos(),sin() fncs each commute with complex-conjugation, i.e

7.5: $\overline{\exp(z)}=\exp(\overline{z})$, $\overline{\cos(z)}=\cos(\overline{z})$, $\overline{\sin(z)}=\sin(\overline{z})$;

Translation-identities & addition-identities

7.6:
$$\cos(z - \tfrac{\pi}{2}) = \sin(z), \quad \sin(z + \tfrac{\pi}{2}) = \cos(z),$$
$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta),$$
$$\sin(\alpha \pm \beta) = \cos(\alpha)\sin(\beta) \pm \sin(\alpha)\cos(\beta).$$

extend to the complex plane. Finally,

7.7: *$\operatorname{Range}(\exp) = \mathbb{C}\setminus\{0\}$ is the punctured $\mathbb{C}$. And $\operatorname{Range}(\cos) = \mathbb{C} = \operatorname{Range}(\sin)$.*

7.8: *All zeros of [complex] cos() lie in $\mathbb{R}$. Hence cos() has only one period, that of 2π. Both statements hold for sin().* ◊

Pf of (7.7). For $\operatorname{Range}(\cos) \overset{?}{=} \mathbb{C}$, target $\frac{\tau}{2}\in\mathbb{C}$ requires z with $\cos(z) = \tau/2$. With $R := e^{iz}$, then, we need $R + \frac{1}{R} = \tau$, i.e $R^2 - \tau R + 1 = 0$. This quad.eqn *has* a solution $R \in \mathbb{C}$. As $R{=}0$ is not a soln, necessarily $R \in \operatorname{Range}(\exp)$. ♦

Pf of (7.8). Fix a $z = x + iy$ st. $\cos(z) = 0$. Thus

$$\begin{aligned} 0 = 2\cos(z) &= \exp(i \cdot [x+iy]) + \exp(-i \cdot [x+iy]) \\ &= \exp(-y + ix) + \exp(y - ix) \\ &= e^{-y}\operatorname{cis}(x) + e^{y}\operatorname{cis}(-x). \end{aligned}$$

Since these summands cancel, they must have equal abs.values. Since x and y are real, then,

$$*: \quad e^{-y} = e^{-y} \cdot |\operatorname{cis}(x)| = e^{y} \cdot |\operatorname{cis}(-x)| = e^{y}.$$

But $\mathbb{R}$-exp() is 1-to-1, so $(*)$ implies that $-y = y$. Hence $y = 0$, i.e z is real. ♦

7e: Lemma. *Familar derivative relations, $\exp' = \exp$ and $\cos' = -\sin$ and $\sin' = \cos$, continue to hold.* ◊

Same-frequency cosines/sines. Consider a sum of same-frequency cosines

$$h(t) := \sum_{j=1}^{N} A_j \cdot \cos(P_j + F \cdot t),$$

where $A_j \in \mathbb{R}$ is *amplitude*, $P_j \in \mathbb{R}$ is *phase-shift* and $F \in \mathbb{R}$ determines the *frequency*. [Courtesy (7.6), we could include sine fncs in the sum.] We seek a phase-shift $\boldsymbol{\theta}$ and amplitude $\mathbf{R} \geq 0$ so that

$$h(t) = \mathbf{R} \cdot \cos(\boldsymbol{\theta} + Ft).$$

From (7.4), we have that $h(t)$ equals

$$\sum_{j=1}^{N} A_j \cdot \operatorname{Re}(e^{i[P_j + Ft]}) \overset{\text{note}}{=\!=} \operatorname{Re}\Bigl(\sum_{j=1}^{N} A_j \cdot e^{i[P_j + Ft]}\Bigr) = \operatorname{Re}\Bigl(\Bigl[\sum_{j=1}^{N} A_j \cdot e^{iP_j}\Bigr] \cdot e^{iFt}\Bigr).$$

Thus we are led to define $\mathbf{S} \in \mathbb{C}$ and $X,Y \in \mathbb{R}$ by

$$\dagger: \quad \mathbf{S} := \Bigl[\sum_{j=1}^{N} A_j \cdot e^{iP_j}\Bigr] =: X + iY.$$

Since each A_j and P_j is real,

$$X = \sum_{j=1}^{N} A_j \cdot \cos(P_j) \quad \text{and} \quad Y = \sum_{j=1}^{N} A_j \cdot \sin(P_j).$$

7f**:** Same-freq Lemma. *[With notation from above.] Set* $\mathbf{R} := |\mathbf{S}| \overset{note}{=\!=} \sqrt{X^2+Y^2}$.

If $\mathbf{S}=0$, *then* $h()$ *is the zero-fnc; so can set* $\boldsymbol{\theta} := 0$. *Otherwise, if* $X=0$, *then set* $\boldsymbol{\theta}$ *to* $\frac{\pi}{2}$ *or* $-\frac{\pi}{2}$ *as* Y *is positive or negative.*

Otherwise: If $X>0$ *then set* $\boldsymbol{\theta} := \arctan(Y/X)$; *and if* $X<0$ *then set* $\boldsymbol{\theta} := \pi + \arctan(Y/X)$.

With $\mathbf{R}, \boldsymbol{\theta}$ *defined as above*

‡**:** $$\left[\sum_{j=1}^{N} \mathrm{A}_j \cdot \cos(\mathrm{P}_j + \mathrm{F}\cdot t)\right] = \mathbf{R}\cdot\cos(\boldsymbol{\theta} + \mathrm{F}t).$$ ◊

7g**:** *E.g.* Compute reals $\mathbf{R} \geq 0$ and phase-shift $\boldsymbol{\theta}$ st.
$$\mathbf{R}\cos(\boldsymbol{\theta}+8t) = \cos(\tfrac{\pi}{3}+8t) + \cos(\tfrac{5\pi}{3}+8t) - \sqrt{2}\cos(\tfrac{7\pi}{4}+8t).$$

SOLN: Applying (†), above,

$$\mathbf{S} = \mathrm{e}^{\mathbf{i}\frac{\pi}{3}} + \mathrm{e}^{\mathbf{i}\frac{5\pi}{3}} - \sqrt{2}\mathrm{e}^{\mathbf{i}\frac{7\pi}{4}} \overset{\text{Geometry}}{=\!=\!=} \mathbf{i}\,.$$

Hence $\mathbf{R} = |\mathbf{i}| = 1$ and $\boldsymbol{\theta} = \operatorname{Arg}(\mathbf{i}) = \frac{\pi}{2}$. □

CCLDE Algorithm [Const.-Coeff LDE]

Initially, we only handle the [*target* = *zero-fnc*] case.

Step S0. Consider numbers $C_0, \ldots, C_N$ and U.F $y=y(t)$ satisfying

$$*: \quad C_N y^{(N)} + C_{N-1} y^{(N-1)} + \ldots + C_1 y' + C_0 y \ = \ 0\,,$$

with $C_N \neq 0$. Define the ***auxiliary polynomial***

$$q(z) \ := \ C_N z^N + C_{N-1} z^{N-1} + \ldots + C_1 z^1 + C_0 z^0\,.$$

We can now re-write $(*)$ as

$$\text{8a:} \qquad [q(\mathbf{D})](y) \ = \ 0\,.$$

Step S1. Let $\mathfrak{R}$ denote the set of **distinct** roots [i.e, zeros] of $q()$. For each root $\mathbf{r} \in \mathfrak{R}$, let $M_{\mathbf{r}} \in \mathbb{Z}_+$ denote the *multiplicity* of $\mathbf{r}$ in $q()$. Thus $\sum_{\mathbf{r}\in\mathfrak{R}} M_{\mathbf{r}}$ equals N, i.e, $\operatorname{Deg}(q)$.

The above says that our polynomial factors as

$$\text{8b:} \qquad q(z) \ = \ C_N \cdot \prod_{\mathbf{r}\in\mathfrak{R}} [z - \mathbf{r}]^{M_{\mathbf{r}}}\,.$$

Step S2. The general solution to (8a) is

$$\text{8c:} \qquad y(t) \ = \ \sum_{\mathbf{r}\in\mathfrak{R}} \ \sum_{j\in[0..M_{\mathbf{r}})} \left[\lambda_{\mathbf{r},j} \cdot t^j \cdot e^{\mathbf{r}\cdot t}\right],$$

freely choosing the N many numbers, $\{\lambda_{\mathbf{r},j}\}_{\mathbf{r},j}$.

Step S3. Now suppose we were given initial conditions, e.g, given specified numbers for values $y(0), y'(0), y''(0), \ldots, y^{(N-1)}(0)$. Or perhaps we are given the value of y'' at N different points.

Differentiate (8c) appropriately and plug in the given points to obtain N equations ["high school" linear equations] which you solve for the values of the N many unknowns $\{\lambda_{\mathbf{r},j}\}_{\mathbf{r},j}$.

CCLDE Example. U.F. $y = y(t)$ satisfies DE

$$y^{(5)} - 6y^{(4)} + 9y^{(3)} + 10y'' - 36y' + 24y \ = \ 0\,.$$

Define $p(z) := z^5 - 6z^4 + 9z^3 + 10z^2 - 36z^1 + 24z^0$; the aux-poly of the above DE. We can re-write the DE as

$$\text{8a}\dagger: \qquad [p(\mathbf{D})](y) \ = \ 0\,.$$

Step S1. Factor polynomial p as

$$\text{8b}\dagger: \quad \begin{aligned} p(z) \ &= \ [z^2 - 3] \cdot [z-2]^3 \\ &= \ [z - U] \cdot [z - V] \cdot [z-2]^3\,, \end{aligned}$$

where $U := \sqrt{3}$ and $V := \text{-}U$. I.e, $\mathfrak{R} = \{U, V, 2\}$ and $M_U = 1$, $M_V = 1$ and $M_2 = 3$.

Step S2. For five arbitrary [possibly complex] numbers $\alpha, \beta, \lambda_0, \lambda_1, \lambda_2$, the function

$$\text{8c}\dagger: \qquad y(t) \ := \ \alpha e^{Ut} + \beta e^{\text{-}Ut} + \left[\sum_{j=0}^{2} \lambda_j \cdot t^j e^{2t}\right]$$

is the general soln to (8a†).

Step S3. Consider IVP (8a†) with

$$y(0) = 2; \quad y'(0) = 0; \quad y''(0) = 4; \quad y^{(3)}(0) = \text{-}12; \quad y^{(4)}(0) = \text{-}30\,.$$

Solving for the coefficients in (8c†) gives

$$\text{8d:} \qquad \alpha = \beta = 1; \quad \lambda_0 = \lambda_1 = 0; \quad \lambda_2 = \text{-}1\,.$$

Consequently, the soln to this IVP is

$$\text{8e:} \qquad y(t) \ = \ \left[e^{\sqrt{3}\cdot t}\right] + \left[e^{\text{-}\sqrt{3}\cdot t}\right] - \left[t^2 e^{2t}\right].$$

***Complex-root* Example.** Your experiments with fluid-flow[♡2] produce U.F. $f = f(t)$ such that

$$\text{8f:} \qquad f''' - [2+\mathbf{i}]f'' + [1+4\mathbf{i}]f' + [2-\mathbf{i}]f \ = \ 0\,.$$

Defining the auxiliary polynomial, then factoring, gives

$$\text{8g:} \quad \begin{aligned} q(z) \ &:= \ z^3 - [2+\mathbf{i}]z^2 + [1+4\mathbf{i}]z + [2-\mathbf{i}] \\ &= \ \left[z - \mathbf{i}\right]^2 \cdot \left[z - [2-\mathbf{i}]\right]. \end{aligned}$$

The solns, $f(t)$, to (8f) are the linear-combinations of

$$e^{\mathbf{i}t}, \ te^{\mathbf{i}t}, \ e^{[2-\mathbf{i}]t}\,.$$

If desired, write $e^{[2-\mathbf{i}]t}$ as $e^{2t} \cdot [\cos(t) - \mathbf{i}\sin(t)]$, since cos() is an even-fnc and sin() an odd-fnc.

[♡2] Wine, with a Milk chaser…

Polynomial target UNDETERMINED COEFFS

[In **NSS9** §4.4, "Undetermined coeffs.".] We study DE

$$\mathsf{V}(f) = G\,, \quad \textit{where the target-poly is}$$

9a: $$G(t) = \sum_{j=0}^{K} B_j t^j\,. \quad \textit{Write } \mathsf{V} = q(\mathbf{D}) \textit{ using aux-poly}$$

$$q(z) = \sum_{n=L}^{N} C_n z^n\,,$$

for natnums $L \leq N$ with $C_L \neq 0$ and $C_N \neq 0$.

Since CCLDOp V() carries polys to polys, we can solve for the coeffs of f. Write a candidate soln as

9b: $$f(t) = \sum_{j=L}^{K+L} u_j \cdot t^j\,,$$

for undetermined numbers $\vec{\mathbf{u}} = (u_0, u_1, \ldots, u_K)$. Equating coeffs in $\mathsf{V}(f) = G$ gives $K+1$ "high school" [e.g, linear] eqns in the $K+1$ unknowns $\vec{\mathbf{u}}$. This system will have (exercise!) a unique soln.

Polynomial-Target 1. U.Poly $f = f(t)$ satisfies

9a†: $$f'' + 5f' + 4f = 8t + 22\,. \quad \textit{So } \mathsf{V} := q(\mathbf{D}), \textit{ where}$$
$$q(z) := z^2 + 5z + 4 = [z - \text{-}4] \cdot [z - \text{-}1]\,.$$

Hence $L = 0$ and $N = 2$. Our target $G(t) := 8t + 22$ has degree $K{=}1$. So poly f has form

9b†: $$f(t) = wt + u$$

for undetermined *numbers* w,u. Thus

$$\mathsf{V}(f) \overset{\text{Why?}}{=\!=} [5\mathbf{D} + 4\mathbf{I}](f) = 4wt + [5w + 4u]\,.$$

[*Why? Did you detect that* $\mathbf{D}^2(f) = 0$ *?*]

Set $4wt + [5w{+}4u]$ equal to target, $8t + 22$, giving eqns $4w = 8$ and $5w + 4u = 22$. Reading L-to-R, $w{=}2$ and $u{=}3$. I.e, $f := 2t + 3$ is sent by V() to G.

(P-T 1 continued) IVP. Mystery fnc $h{=}h(t)$ satisfies

∗1: $$h'' + 5h' + 4h = 8t + 22\,, \quad \text{together with}$$
∗2: $$h(0) = 0 \quad \text{and} \quad h'(0) = \text{-}1\,.$$

From (9a†), we know that $\mathrm{e}^{\text{-}4t}$ and $\mathrm{e}^{\text{-}t}$ are each mapped to 0 by V(). Consequently, the general soln, h, to (∗1) has form

$$h(t) = \alpha\mathrm{e}^{\text{-}4t} + \beta\mathrm{e}^{\text{-}t} + [2t + 3]\,,$$

for constants α,β. Eqns (∗2) yield $\alpha = 2$ and $\beta = \text{-}5$. Thus fnc

$$h(t) = 2\mathrm{e}^{\text{-}4t} - 5\mathrm{e}^{\text{-}t} + [2t + 3]$$

is the unique soln to Mystery-IVP (∗1, ∗2).

Polynomial-Target 2. U.Poly $f = f(t)$ satisfies

9a‡: $$f'' + 3f' = 9t^2 + 6t - 3\,. \quad \textit{So } \mathsf{V} := 3q(\mathbf{D}), \textit{ where}$$
$$q(z) := z^2 + 3z = [z - 0] \cdot [z - \text{-}3]\,.$$

Hence $L = 1$ and $N = 2$. Target $G(t) := 9t^2 + 6t - 3$ has degree $K{=}2$. Thus polynomial f has form

9b‡: $$f(t) = wt^3 + vt^2 + ut$$

for not-yet-determined numbers w,v,u. Computing,

$$\mathsf{V}(f) = f'' + 3f' = 9wt^2 + [6w{+}6v]t + [2v{+}3u]\,.$$

Equating coeffs with $G := 9t^2 + 6t - 3$ produces

$$9w = 9 \quad \textit{and} \quad 6w + 6v = 6 \quad \textit{and} \quad 2v + 3u = \text{-}3\,.$$

Hence $w = 1$, so $v = 0$, thus $u = \text{-}1$.

THE UPSHOT: Function $f := t^3 - t$ is sent by V() to G. Consequently, the general (9a‡)-solution is $f_{\alpha,\beta}(t) = \alpha + \beta\mathrm{e}^{\text{-}3t} + [t^3 - t]$.

P-T 2, alternative. Fnc $h := f'$ satisfies $h' + 3h = G$. Since $\mathrm{Deg}(h) = \mathrm{Deg}(G) = 2$; our $h = Pt^2 + Qt + R$, for some numbers P,Q,R. Consequently,

$$9t^2 + 6t - 3 \overset{\text{by DE}}{=\!=} [\mathbf{D} + 3\mathbf{I}](h) = 3Pt^2 + [2P + 3Q]t + 3R\,.$$

Hence $9 = 3P$; so $\boxed{P = 3}$. And $6 = 2P + 3Q = 6 + 3Q$; thus $\boxed{Q = 0}$. Lastly, $\text{-}3 = 3R$, whence $\boxed{R = \text{-}1}$. THE UPSHOT IS. . .

$$f \overset{\text{def}}{=} \int h = \int [3t^2 - 1]\,\mathrm{d}t = t^3 - t\,,$$

as before.

PolyExp target

A ***PolyExp*** is a poly×exponential; e.g $F(t) := [3+t^2]\cdot e^{4t}$.

Step P0. Consider a CCLDOp $L()$ and DE

10a: $$L(y) = G_1(t)\,e^{M_1 t} + G_2(t)\,e^{M_2 t} + \ldots,$$

where each G_j is a polynomial and each ***exponent-Multiplier*** M_j is a number. For each polyExp, we will compute a fnc y_j s.t $L(y_j) = G_j\cdot e^{M_j t}$. Then $y_1 + y_2 + \ldots$ is a particular soln to (10a). Adding the gen.soln z to $L(z) = 0$ gives the gen.soln to (10a), *since* L *is linear*.

We've reduced the problem to solving DEs OTForm $L(y) = G\cdot e^{Mt}$, where G is a poly. We'll compute a soln OTForm $y := f\cdot e^{Mt}$, where f is a poly.

Step P1. For an arb.fnc f and arb.number μ, let $E := e^{\mu t}$ and note that $E' = \mu E$. Compute $L(fE)$ to produce a CCLDOp V_μ, that depends on the number μ, such that
$$L(f\cdot E) = V_\mu(f)\cdot E.$$

So $L()$ sends $f\cdot e^{Mt}$ to $G\cdot e^{Mt}$ IFF f satisfies
$$V_M(f) = G.$$

Use (9a), UNDETERMINED COEFFS, to solve for f.

Defn. Call $V_\mu = V_{L,\mu}$ the operator ***"associated to*** *operator* L *and number* μ***"***. □

Preliminary computation. To speed up our numerical example, let's pre-compute the L-to-V_μ transition for a general quadratic CCLDOp $L()$.

Numbers $\mathbf{r}_1$,$\mathbf{r}_2$ yield a quadratic poly
$$q(z) := [z-\mathbf{r}_1][z-\mathbf{r}_2] = z^2 - \mathcal{S}z + \mathcal{P},$$

using the sum $\mathcal{S} := \mathbf{r}_1 + \mathbf{r}_2$, and product $\mathcal{P} := \mathbf{r}_1\mathbf{r}_2$, of the roots. The corresponding operator is
$$L(y) := y'' - \mathcal{S}y' + \mathcal{P}y.$$

For arb. number μ and fnc f, letting $E := e^{\mu t}$, note

$$[f\cdot E]^{(0)} = f\cdot E;$$
$$[f\cdot E]^{(1)} = f'E + fE' \overset{\text{note}}{=\!=} [f' + \mu f]\cdot E;$$
$$[f\cdot E]^{(2)} = [f'' + \mu f']E + [f' + \mu f]\cdot\mu E = [f'' + 2\mu f' + \mu^2 f]\cdot E.$$

Consequently, $L(f\cdot E) = V_\mu(f)\cdot E$ where

10b: $$V_\mu(f) = f'' + [2\mu - \mathcal{S}]f' + [\mu^2 - \mathcal{S}\mu + \mathcal{P}]f \overset{\text{note}}{=\!=} f'' + [2\mu - \mathcal{S}]f' + [q(\mu)]f.$$

[The coeff of f will always be $q(\mu)$.]

***PolyExp-target* Example 1.** Consider DE

10a†: $$y'' - y' - 2y = \overbrace{[8t+22]e^{3t}}^{\mathcal{A}} + \overbrace{[9t^2+6t-3]e^{2t}}^{\mathcal{B}}.$$

Hence $\mathcal{S} = 1$ and $\mathcal{P} = -2$, *and*
$$q(z) = z^2 - z - 2 = [z+1]\cdot[z-2].$$

Thus $\mathbf{r}_1 = -1$ and $\mathbf{r}_2 = 2$. Courtesy (10b),
$$V_\mu(f) = f'' + [2\mu - 1]f' + [\mu^2 - \mu - 2]f.$$

Let's compute fncs y_a and y_b so that $L(y_a) = \mathcal{A}$ and $L(y_b) = \mathcal{B}$, recalling that (10a†) defined $\mathcal{A}$ and $\mathcal{B}$.

PolyExp $\mathcal{A}$**.** Note $V_3(f) = f'' + 5f' + 4f$. With $G := 8t + 22$, then, we seek f such that $V_3(f) = G$. Happily, (9a†) solved this; set $y_a := [2t+3]\cdot e^{3t}$.

PolyExp $\mathcal{B}$**.** Observe $V_2(f) = f'' + 3f'$. Setting $G := 9t^2 + 6t - 3$, we seek f for which $V_2(f) = G$. *A Stroke of Good Fortune!* –example (9a‡) to the rescue. We can let $y_b := [t^3 - t]\cdot e^{2t}$.

Assembling the pieces. Our hard work has paid off. Recalling roots $\mathbf{r}_1$ and $\mathbf{r}_2$, the (10a†) gen.soln is
$$y_{\alpha,\beta}(t) = \alpha\,e^{-t} + \beta\,e^{2t} + [2t+3]\cdot e^{3t} + [t^3 - t]\cdot e^{2t}$$
$$= \alpha e^{-t} + [2t+3]\cdot e^{3t} + [t^3 - t + \beta]\cdot e^{2t}.$$

Nifty! Worth the price of admission...

***PolyExp-tar* Ex. 2.** For poly $q(z) := z^2 - 7z + 12$ and operator $\mathsf{R} := q(\mathbf{D})$, consider DE

$$\text{10a‡:} \qquad \mathsf{R}(y) = \overbrace{6\mathrm{e}^{2t}}^{\mathcal{A}} + \overbrace{[2t-2]\mathrm{e}^{3t}}^{\mathcal{B}} + \overbrace{60}^{\mathfrak{C}} .$$

Aux-poly $q()$ has root-sum $\mathcal{S} := 7$ and root-product $\mathcal{P} := 12$. So (10b) tells us, for each number μ, that the *associated operator* $\mathsf{V}_\mu = \mathsf{V}_{\mathsf{R},\mu}$ is

$$*: \qquad \mathsf{V}_\mu(f) = f'' + [2\mu - 7]f' + [q(\mu)]f .$$

Let's compute fncs y_a, y_b and y_c so that $\mathsf{R}(y_a) = \mathcal{A}$, $\mathsf{R}(y_b) = \mathfrak{B}$, and $\mathsf{R}(y_c) = \mathfrak{C}$, from (10a‡).

For future reference, note $q()$ factors as

$$\text{¥:} \qquad q(z) = [z-3]\cdot[z-4] .$$

So e^{3t} and e^{4t} are annihilated by $\mathsf{R}()$.

PExp $\mathcal{A}$**.** From (∗,¥), our $\mathsf{V}_2(f) = f'' - 3f' + 2f$. We seek f s.t $\mathsf{V}_2(f) = 6$; so f must have degree zero. Writing $f() := w$, we see that $6 = \mathsf{V}_2(f) = 2w$; thus $w = 3$. *I.e,* $y_a := 3\mathrm{e}^{2t}$ *is sent by* $\mathsf{R}()$ *to* $\mathcal{A}$.

PExp $\mathfrak{B}$**.** Courtesy (∗,¥), our $\mathsf{V}_3(f) = f'' - f'$. A polynomial f s.t $\mathsf{V}_3(f) = 2t - 2$ has form

$$f(t) = wt^2 + vt .$$

Computing, $\mathsf{V}_3(f) = \text{-}2wt + [2w - v]$. Setting this equal to $2t - 2$ gives $w = \text{-}1$ and $v = 0$. *Consequently,* $y_b := -t^2\mathrm{e}^{3t}$ *is sent by* $\mathsf{R}()$ *to* $\mathfrak{B}$.

PExp $\mathfrak{C}$**.** Note that $\mathfrak{C} = 60\mathrm{e}^{0\cdot t}$. Our (∗,¥) says $\mathsf{V}_0(f) = f'' - 7f' + 12f$; i.e, $\mathsf{V}_0 = \mathsf{R}$, as it must. [*Why?*]

What polynomial f has $\mathsf{V}_0(f) = 60$? Why $f() = \frac{60}{12} = 5$, of course! Unsurprisingly, $y_c := 5$ is sent by $\mathsf{R}()$ to $\mathfrak{C}$.

Assembly. Recalling roots 3 and 4 of our aux-poly (¥), the general-soln to (10a‡) is

$$\text{10b:} \quad y_{\alpha,\beta}(t) = \alpha\mathrm{e}^{3t} + \beta\mathrm{e}^{4t} + 3\cdot\mathrm{e}^{2t} - t^2\cdot\mathrm{e}^{3t} + 5 .$$

Terms can be combined, if desired. *Copasetic!*

***PolyExp-tar* Ex. 3.** For poly $q(z) := z^3 - 3z^2 + 5$ and operator $\mathsf{P} := q(\mathbf{D})$, consider DE

$$\text{10a¥:} \qquad \mathsf{P}(y) = t^2\mathrm{e}^{2t} .$$

For arb. fnc f, letting $E := \mathrm{e}^{2t}$, note

$$[f\cdot E]^{(0)} = f \cdot E ;$$
$$[f\cdot E]^{(1)} = f'E + fE' \stackrel{\text{note}}{=\!=} [f' + 2f]\cdot E ;$$
$$[f\cdot E]^{(2)} = [f'' + 2f']E + [f' + 2f]\cdot 2E = [f'' + 4f' + 4f]\cdot E ;$$
$$[f\cdot E]^{(3)} = [f''' + 4f'' + 4f']E + [2f'' + 8f' + 8f]E = [f''' + 6f'' + 12f' + 8f]\cdot E .$$

Recall that the *associated operator* $\mathsf{V} = \mathsf{V}_{\mathsf{P},2}$ is defined by $\boxed{\mathsf{P}(f\cdot E) = \mathsf{V}(f)\cdot E}$. So

$$\text{10c:} \quad \mathsf{V}(f) = f''' + [6-3]f'' + [12-12]f' + [8-12+5]f \stackrel{\text{note}}{=\!=} f''' + 3f'' + f .$$

[As it must, the coeff of f is $q(2)$.]

We seek a poly f solving $\mathsf{V}(f) = t^2$, so write

$$f = wt^2 + vt + u . \quad \textit{Note } f''' = \mathbf{0}. \textit{ Hence}$$
$$\mathsf{V}(f) = wt^2 + vt + [6w + u] \stackrel{\text{Goal}}{=\!=} t^2 .$$

Solving, $w = 1$ and $v = 0$ and $u = \text{-}6$.

So $y(t) := [t^2 - 6]\mathrm{e}^{2t}$ is <u>a</u> soln to (10a¥). However, the gen.soln is harder to obtain, as computing the roots of the above $q()$ is not so easy. [Cardano's formula can be used.]

Linear maps

A *vector space* is like $\mathbb{R}\times\mathbb{R}$ [or $\mathbb{C}\times\mathbb{C}$] with component-wise addition: For vectors $\mathbf{v}_j := (x_j, y_j)$, their *sum* $\mathbf{v}_1 + \mathbf{v}_2$ is $(x_1 + x_2,\ y_1 + y_2)$. More generally, a ***vector space***[♡3] is a set $\mathbf{V}$ (or it might be called $\mathbf{W}$ or $\mathbf{E}$ or $\mathbf{H}$ or...) together with an addition which is *commutative* and *associative*. Also, we can multiply a vector by a ***scalar*** which is either a real number or, more generally, a complex number.

So a VS is a tuple $(\mathbf{V}, +, \mathbf{0}, \cdot, \mathbb{R})$ when the scalars are reals, or $(\mathbf{V}, +, \mathbf{0}, \cdot, \mathbb{C})$ when we allow complex scalars.

11a: *Defn.* Now consider a map $\mathsf{L}\colon \mathbf{V} \to \mathbf{W}$ between vector spaces $(\mathbf{V}, +, \mathbf{0}, \cdot, \mathbb{C})$ and $(\mathbf{W}, +, \mathbf{0}, \cdot, \mathbb{C})$. This map L is ***linear*** IFF:

$\forall\, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v} \in \mathbf{V}$ and for all scalars α, our L satisfies

£1: $\mathsf{L}(\mathbf{v}_1 + \mathbf{v}_2) = \mathsf{L}(\mathbf{v}_1) + \mathsf{L}(\mathbf{v}_2)$ *and*

£2: $\mathsf{L}(\alpha \cdot \mathbf{v}) = \alpha \cdot \mathsf{L}(\mathbf{v})$.

Equivalently: *For all vectors $\mathbf{v}_1, \ldots, \mathbf{v}_N$ and for all scalars $\alpha_1, \ldots, \alpha_N$:*

$$\mathsf{L}\Big(\sum_{j=1}^{N} \alpha_j \mathbf{v}_j\Big) = \sum_{j=1}^{N} \alpha_j \mathsf{L}(\mathbf{v}_j)\,. \qquad \Box$$

11b: *Span Defn.* The <u>set</u> of all linear-combinations [***lin-combs***] of a collection $\mathcal{S} := \{\mathbf{v}_1, \ldots, \mathbf{v}_N\}$ of vectors is called "the ***span*** of $\mathcal{S}$". I.e, Span($\mathcal{S}$) equals

$$\operatorname{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_N) := \left\{ \sum_{j=1}^{N} \alpha_j \mathbf{v}_j \;\middle|\; \begin{array}{l}\text{Where } \alpha_1, \ldots, \alpha_N \\ \text{are scalars.}\end{array} \right\}.$$

Our $\mathcal{S}$ <u>is</u> a ***linearly-independent set***[an ***L.I-set***] if the <u>**only**</u> list $\beta_1, \ldots, \beta_N$ of scalars satisfying

$$\Big[\sum_{j=1}^{N} \beta_j \mathbf{v}_j\Big] = \mathbf{0} \quad \text{[the zero vector]} \ \Box$$

is $\beta_1 = 0$, $\beta_2 = 0 \ldots, \beta_N = 0$. [See §6.1–**NSS9** P.323, & §4.2]

VS examples. For N a natnum or ∞, let Diff^N be the VS of N-times differentiable fncs, with $\mathbf{C}^N \subset \mathsf{Diff}^N$ the sub-VS of fncs whose N^{th}-derivative is cts. So

$$\mathsf{Diff}^0 \supsetneqq \mathbf{C}^0 \supsetneqq \mathsf{Diff}^1 \supsetneqq \mathbf{C}^1 \supsetneqq \mathsf{Diff}^2 \supsetneqq \ldots \supsetneqq \mathsf{Diff}^\infty \overset{\text{note}}{\equiv} \mathbf{C}^\infty.$$

E.g, fnc $|x|$ is in $\mathbf{C}^0$, the space of cts fncs, but is not in Diff^1, since abs.value is not differentiable at the origin. N.B: Often $\mathbf{C}$ is written for $\mathbf{C}^0$, the cts fncs. $\Box$

[♡3] Abbreviate 'vector space' as VS, and 'vector spaces' as VSes.

Conjugate-root example

A polynomial with all real coeffs [a "real-poly" or "$\mathbb{R}$-poly"] factors into a product of $\mathbb{R}$-irreducible linear and quadratic real-polys.

The ***discriminant*** of quadratic [i.e, $A \neq 0$] polynomial $q(z) := Az^2 + Bz + C$ is

12.1: $$\operatorname{Discr}(q) := B^2 - 4AC\,,$$

and its zeros ["roots"] are

12.2: $$\frac{1}{2A}\Big[-B \pm \sqrt{\operatorname{Discr}(q)}\,\Big].$$

When A,B,C are real, then, the non-real zeros of q come in complex-conjugates pairs.

13: Same-span Lemma. *Here,* Span *means* $\mathbb{C}$-Span. *Fix J,K complex numbers [usually real, in practice]. Then*

$$\operatorname{Span}\Big(\mathsf{e}^{[J+\mathbf{i}K]t}, \mathsf{e}^{[J-\mathbf{i}K]t}\Big) = \operatorname{Span}\Big(\mathsf{e}^{Jt}\cdot\cos(Kt)\,,\ \mathsf{e}^{Jt}\cdot\sin(Kt)\Big) \overset{\text{note}}{=\!=} \mathsf{e}^{Jt}\cdot\operatorname{Span}\Big(\cos(Kt)\,,\ \sin(Kt)\Big)\,.$$

Indeed, for numbers α, β, μ, ν, we have

13a: $$\alpha\cdot\mathsf{e}^{[J+\mathbf{i}K]t} + \beta\cdot\mathsf{e}^{[J-\mathbf{i}K]t} \quad \textit{equals} \quad \mathsf{e}^{Jt}\cdot[\mu\cdot\cos(Kt) + \nu\cdot\sin(Kt)]\,,$$

where the scalars are related by

13b: $$\mu = \alpha + \beta \quad \textit{and} \quad \nu = \mathbf{i}\cdot[\alpha - \beta]\,;$$

13c: $$\alpha = \frac{\mu - \mathbf{i}\cdot\nu}{2} \quad \textit{and} \quad \beta = \frac{\mu + \mathbf{i}\cdot\nu}{2}\,. \qquad \Diamond$$

Proof. Lemma (7c) and routine algebra. ♦

Eric's requested IVP (Reverse engineering).

Let's create a CCLDE whose diff-operator polynomial $q()$ has a specified complex-conjugate roots; say $U := 3+2\mathbf{i}$ and $\overline{U} = 3-2\mathbf{i}$. Define

$$*: \quad q(z) := [z - U]\cdot[z - \overline{U}] \overset{\text{note}}{=\!=} z^2 - 6z + 13\,.$$

Let's go through the steps to solve DE

$$**: \qquad f'' - 6f' + 13f = 0$$

with initial conditions $f(0)=1$ and $f'(0)=13$.

By CCLDE, the soln-set to $(**)$ is $\mathbb{C}$-Spn$(\mathsf{e}^{Ut}, \mathsf{e}^{\overline{U}t})$. To re-write using cos() and sin(), define expressions

$$E := \mathsf{e}^{3t},\ \ C := \cos(2t),\ \ S := \sin(2t)\,.$$

Courtesy (13a), there are numbers μ, ν so that

$$f(t) := E\cdot[\mu C + \nu S]$$

satisfies the initial conditions. This gives

$$1 = f(0) = 1\cdot[\mu\cdot 1 + \nu\cdot 0] \overset{\text{note}}{=\!=} \mu\,.$$

Diff'ing gives $f'(t) = 3E\cdot[\mu C+\nu S] + E\cdot[\text{-}2\mu S + 2\nu C]$. So $13 = f'(0)$, which equals

$$\begin{aligned} 3\cdot[\mu+0] + 1\cdot[\text{-}2\mu\cdot 0 + 2\nu\cdot 1] &= 3\mu + 2\nu \\ &= 3 + 2\nu\,. \end{aligned}$$

Hence $\nu = 5$. Thus the soln to the IVP is

$$*1: \quad f(t) = \mathsf{e}^{3t}\cdot[\cos(2t) + 5\sin(2t)]$$

$$*2: \quad \overset{\text{by (13c)}}{=\!=} \frac{1-5\mathbf{i}}{2}\cdot\mathsf{e}^{[3+2\mathbf{i}]t} + \frac{1+5\mathbf{i}}{2}\cdot\mathsf{e}^{[3-2\mathbf{i}]t}\,.$$

Prefer a single trig-fnc with phase shift? Easily,

$$\begin{aligned} \cos(2t) + 5\sin(2t) &= \cos(2t) + 5\cos(2t - \tfrac{\pi}{2}) \\ &\overset{\text{by Same-freq (7f)}}{=\!=} \mathbf{R}\cdot\cos(\boldsymbol{\theta} + 2t)\,, \end{aligned}$$

where $\mathbf{R} := \sqrt{1^2+5^2} = \sqrt{26} \approx 5.099$,

and $\boldsymbol{\theta} := \arctan(\frac{-5}{1}) = -\arctan(5) \approx \text{-}1.373$. □

Mass-spring [NSS in §4.1, §4.2, §4.9]

Abstract/concrete units. Symbol :: means *"has abstract units of"*. E.g, [Height of Little Hall] :: ⓓ.

ⓓ	in inches	ft, mi feet,miles	cm, m (centi)meters			
ⓣ	sec seconds	min minutes	hr hours	mph=$\frac{\text{mi}}{\text{hr}}$		
ⓓ^{3}	gal gallons	lit liters=1000cm^3		[*volume*]		
ⓜ	kg kilograms			[*mass*]		
ⓦ	lb pounds	oz ounces	$\frac{\text{kg}\cdot\text{m}}{\text{sec}^2}$	[*weight, force*]		
ⓟ	°F Fahrenheit	°C Celsius				
ⓓ̸	*Dimensionless*	ⓘ? *Units depend on application*				

CONVENTION: These notes will typically write zero without units, i.e, 0 rather than 0min or $0\frac{\text{kg}\cdot\text{m}}{\text{sec}^2}$.

Harmonic motion. Our parameters are

M :: ⓜ	Mass of object. [>0]
B :: ⓜ/ⓣ	Damping coefficient. [≥0]
K :: ⓜ/ⓣ^{2}	Hooke's constant of the spring. [>0]
$y{=}y(t)$ :: ⓓ	Position of the mass at time t.
ω :: 1/ⓣ	(Angular) frequency, $\frac{\text{radians}}{\text{time}}$.

An unforced spring has DE

14: $\quad \mathsf{M}y'' + \mathsf{B}y' + \mathsf{K}y = 0\,.$ Here, let 0 implicitly take on units of force.

The corresponding aux-poly is

$$q(z) := \mathsf{M}z^2 + \mathsf{B}z + \mathsf{K}, \quad \textit{with}$$
$$\Delta := \operatorname{Discr}(q) = \mathsf{B}^2 - 4\mathsf{MK} :: \left[\frac{ⓜ}{ⓣ}\right]^2 \quad \textit{and}$$
$$\operatorname{Roots}(q) = \frac{-\mathsf{B}}{2\mathsf{M}} \pm \frac{\sqrt{\Delta}}{2\mathsf{M}} = \frac{-\mathsf{B}}{2\mathsf{M}} \pm \sqrt{\left[\frac{\mathsf{B}}{2\mathsf{M}}\right]^2 - \frac{\mathsf{K}}{\mathsf{M}}} :: \frac{1}{ⓣ}.$$

CASE: $\Delta < 0$, underdamped Set

$$\omega := \frac{\sqrt{-\Delta}}{2\mathsf{M}} \quad \textit{and} \quad R := \frac{\mathsf{B}}{2\mathsf{M}}. \qquad \textit{So}$$
$$\operatorname{Roots}(q) = -R \pm \mathbf{i}\omega\,.$$

Thus the soln-set to (14) is

$$\mathsf{e}^{-Rt}\cdot\operatorname{Span}\Big(\cos(\omega t),\ \sin(\omega t)\Big)$$
$$= \mathsf{e}^{-Rt}\cdot\operatorname{Span}\Big(\mathsf{e}^{\mathbf{i}\omega t},\ \mathsf{e}^{-\mathbf{i}\omega t}\Big)$$
$$= \operatorname{Span}\Big(\mathsf{e}^{[-R+\mathbf{i}\omega]t},\ \mathsf{e}^{[-R-\mathbf{i}\omega]t}\Big)\,.$$

CASE: $\Delta = 0$, critically damped Aux-poly has one real root, negative, of multiplicity 2. Etc.

CASE: $\Delta > 0$, overdamped Aux-poly has two (distinct) negative real roots. Etc.

Viewing* M *and* K *as fixed. The ***natural*** undamped, $\mathsf{B} = 0$, frequency is $\omega_{\text{Nat}} = \sqrt{\frac{\mathsf{K}}{\mathsf{M}}}$. The ***critical-damping coeff*** is $\mathsf{B} := 2\sqrt{\mathsf{MK}}$.

Pendulum. Consider a length L :: ⓓ pendulum, under a uniform acceleration [gravitional] field A :: $\frac{ⓓ}{ⓣ^2}$. Let $\theta{=}\theta(t)$ denote its angle w.r.t vertical. At time t, the observed acceleration of the bob is $\mathsf{L}\cdot\theta''(t)$, whereas the acceleration from A is $-\mathsf{A}\cdot\sin(\theta(t))$, giving DE

15: $$\theta'' = -\frac{\mathsf{A}}{\mathsf{L}}\cdot\sin(\theta)\,.$$

If the max-value of $\theta()$ *is small*, then we can use approximation $\frac{\sin(\theta)}{\theta} \approx 1$ to get approximating DE

16a: $$\theta'' = -\frac{\mathsf{A}}{\mathsf{L}}\cdot\theta\,.$$

This Harmonic.DE has $\omega := \sqrt{\frac{\mathsf{A}}{\mathsf{L}}} :: \frac{1}{ⓣ}$.

Adjoined paragraph: With θ_0 the time-zero displacement (initial angle), our (16a) has soln

16b: $$\theta(t) = \alpha\sin(\omega\cdot t) + \theta_0\cos(\omega\cdot t), \quad \textit{with angular speed}$$
$$\theta'(t) = [\alpha\cos(\omega\cdot t) - \theta_0\sin(\omega\cdot t)]\cdot\omega\,.$$

The FOLDE algorithm [First-Order LDE]

[§2.3–**NSS9**.]

Step F0. Write the DE in the form

17a: $$\frac{\mathrm{d}y}{\mathrm{d}x} + [C(x)\cdot y] = G(x).$$

Pick [i.e, compute] an antiderivative $B()$ of $C()$, i.e

17b: $$B(x) := \int^x C().$$

For later use, store this *multiplier function*[♡4] M:

17c: $$M(x) := \mathrm{e}^{B(x)} = [\text{simplified}].$$

Observe that $M' = M\cdot C$. Hence

$$\begin{aligned}[M\cdot y]' &= [M\cdot C\cdot y] + [M\cdot y'] \\ &= M\cdot\Big[[C\cdot y] + y'\Big] \\ &\underset{\text{by (17a)}}{=} M\cdot G.\end{aligned}$$

Step F1. Define product $P(x) := M(x)\cdot G(x)$. Compute an antiderivative,

17d: $$Q(x) := \int^x P().$$

Step F2. Now, for $\alpha :=$ [an arbitrary constant], the following definition of y will satisfy equation (17a):

17e: $$y(x) = y_\alpha(x) := \frac{\alpha}{M(x)} + \frac{Q(x)}{M(x)}.$$

Step F3. Use (17e) to compute y'. Plug in to (17a) to see if your formula for y satisfies it. [It is at this point that I sometimes find that I have made a computational error.]

Step F4. If the problem asks that y satisfy –in addition to (17a)– an initial condition of the form $y(x_0) = y_0$, then substitute $x = x_0$ and $y = y_0$ into (17e) and solve for α. You will get that

17f: $$\alpha = [y_0\cdot M(x_0)] - Q(x_0).$$

That's all there is to it! It's all copasetic.

FOLDE Example. Given DE

17a†: $$\begin{aligned} x^3y' + x^2y &= 7x^8 - x^5, \quad \text{re-write it as} \\ y' + \tfrac{1}{x}\cdot y &= 7x^5 - x^2, \end{aligned}$$

to fit form (17a). So $G(x) = [7x^5 - x^2]$.

Applying step (**F0**), we have $C(x) = 1/x$, and can define $B := \log$. Hence

17c†: $$M(x) \overset{\text{def}}{=\!=} \mathrm{e}^{\log(x)} \overset{\text{note}}{=\!=} x.$$

Step F1. Define $P(x) := x\cdot[7x^5 - x^2] = 7x^6 - x^3$. Antidifferentiate to get

17d†: $$Q(x) := x^7 - \tfrac{1}{4}x^4.$$

Step F2. For each constant, α, the function

17e†: $$y_\alpha(x) := \frac{\alpha}{x} + [x^6 - \tfrac{1}{4}x^3]$$

is supposed to satisfy (17a†). **Check that it does!**

Step F4. Imagine we are given initial condition

17g: $$y(2) = 66.5.$$

For the corresponding α, compute

$$y_\alpha(2) = \frac{\alpha}{2} + 64 - 2 = \frac{\alpha}{2} + 62.$$

Hence $\alpha/2 = 66.5 - 62 = 4.5$, so $\boxed{\alpha = 9}$. Alternatively, formula (17f) gives

$$\begin{aligned}\alpha &= [66.5\cdot M(2)] - Q(2) \\ &= [66.5\cdot 2] - [128 - 4] \\ &= 133 - 124 \overset{\text{note}}{=\!=} 9.\end{aligned}$$

THE UPSHOT: The unique soln to IVP (17a†, 17g) is

$$y(x) = [9/x] + x^6 - \tfrac{1}{4}x^3.$$

[♡4] Using functional notation, we could write $M := \exp\circ B$.

FOLDE Trig-Example. U.F. $y{=}y(t)$ satisfies

17a‡: $$y' + \cos(2t)\cdot y = \cos(2t)\,.$$

Applying (**F0**) conveniently hands us

17b‡: $$B(t) := \int^t \cos(2\tau)\,\mathrm{d}\tau \overset{\text{note}}{=\!=} \tfrac{1}{2}\sin(2t)\,.$$

To lessen writing, define *expressions*

$$\mathbf{c} := \cos(2t) \quad \textit{and} \quad \mathbf{s} := \sin(2t)\,.$$

Thus our multiplier is

17c‡: $$M(t) := \mathrm{e}^{\frac{1}{2}\mathbf{s}}\,,$$

and the corresponding product is $P(t) := \mathbf{c}\cdot\mathrm{e}^{\frac{1}{2}\mathbf{s}}$. An antiderivative is

17d‡: $$Q(t) := \mathrm{e}^{\frac{1}{2}\mathbf{s}} \overset{\text{note}}{=\!=} M(t)\,,$$

so $\frac{Q}{M} = 1$.

THE UPSHOT: For each constant α, function

17e‡: $$y(t) = y_\alpha(t) := \alpha\cdot\mathrm{e}^{-\frac{1}{2}\sin(2t)} + 1$$

will satisfy (17a‡).

CHECKING: Note $[\frac{1}{2}\mathbf{s}]' = \mathbf{c}$. Hence differentiating (17e‡) gives

$$y' = \alpha\mathrm{e}^{-\frac{1}{2}\mathbf{s}}\cdot[-\mathbf{c}]\,. \qquad \textit{And}$$
$$\mathbf{c}\cdot y = \alpha\mathrm{e}^{-\frac{1}{2}\mathbf{s}}\cdot\mathbf{c} + \mathbf{c}\,.$$

Their sum is $\boxed{y' + \mathbf{c}\cdot y = \mathbf{c}}$, which indeed is (17a‡).

log-CoV to FOLDE [Change-of-Variable]

Consider a *positive-valued fnc* $y{=}y(t)$ satisfying DE

18a: $$y' - [G(t)\cdot y] = -C(t)\cdot y\cdot\log(y)\,.$$

Happily, we can convert this to a FOLDE, by setting $z := \log(y)$. Divide by y and re-order as

$$[y'/y] + C(t)\log(y) = G(t)\,.$$

Our substitution allows us to re-write this as

18b: $$z' + C(t)\cdot z = G(t)\,,$$

which has form (17a). Its general soln $z_\alpha()$ hands us

18c: $$y_\alpha(t) = \mathrm{e}^{z_\alpha(t)} = \exp(z_\alpha(t))\,.$$

Example of CoV-to-FOLDE. For $t{>}0$, we seek a *positive-valued fnc* $y{=}y(t)$ satisfying

18a†: $$ty' = 2t^2y + [y\cdot\log(y)]\,.$$

Dividing by $t\cdot y$ and re-ordering gives

$$\frac{y'}{y} - [\tfrac{1}{t}\cdot\log(y)] = 2t\,.$$

Substitution $z := \log(y)$ gives

18b†: $$z' - [\tfrac{1}{t}\cdot z] = 2t\,.$$

Matching to (17a), we define

$$G(t) := 2t,\ C(t) := \tfrac{-1}{t},\ B := -\log\,,$$
$$\text{and}\ \ M(t) := \mathrm{e}^{B(t)} = \tfrac{1}{t}\,.$$

Step (**F1**) gives $P(t) := \frac{1}{t}\cdot 2t = 2$, hence $Q(t) := 2t$. For an arbitrary constant α, then,

17e†: $$z_\alpha(t) := \alpha\cdot t + 2t\cdot t\,.$$

"Un-substituting" [returning to y], then, yields

18c†: $$y_\alpha(t) = \mathrm{e}^{\alpha t + 2t^2}\,.$$

Have you checked that this really satisfies (18a†)*?*

Bernoulli eqn using FOLDE

Given fncs $\widetilde{C}$ and $\widetilde{G}$, we seek solutions $y() > 0$ to

19a: $$y' + \widetilde{C}\cdot y = \widetilde{G}\big/ y^{[N-1]},$$

where $N \in \mathbb{R}$ with $N \neq 0$. [*When N is zero, the DE is $y' + \widetilde{C}\cdot y = \widetilde{G}\cdot y$. This rewrites as $y' + [\widetilde{C} - \widetilde{G}]\cdot y = 0$, the easy ZeroTar case of FOLDE.*]

To convert (19a) to a LDE, multiply both sides by $N\cdot y^{[N-1]}$ to get

$$N y^{[N-1]}\cdot y' + N\widetilde{C}\cdot y^N = N\cdot\widetilde{G}.$$

With CoV $z = y^N$, this becomes

19b: $$z' + \underbrace{N\cdot\widetilde{C}(t)}_{C(t)}\cdot z = \underbrace{N\cdot\widetilde{G}(t)}_{G(t)}.$$

Apply the FOLDE algorithm to obtain a general soln z_α. Finally, take the (positive) N^{th}-root to get

19c: $$y_\alpha := [z_\alpha]^{1/N}.$$

Bernoulli eqn Example. U.F. $y{=}y(t)$ has

19a†: $$y' + 2y = t\cdot y^{-2} \stackrel{\text{note}}{=\!=} t\big/ y^{[3-1]}.$$

So $N = 3$ and $\widetilde{C}(t) = 2$ and $\widetilde{G}(t) = t$. Change-of-variable $z := y^3$ gives [via DE $3y^2y' + 6y^3 = 3t$]

19b†: $$z' + 6z = 3t.$$

So $B(t) := 6t$ and $M(t) = \mathrm{e}^{6t}$. Thus product

$$P(t) := M(t)\cdot 3t \stackrel{\text{note}}{=\!=} 3t\cdot\mathrm{e}^{6t}.$$

Courtesy (1.1), one antiderivative of P is

$$Q(t) := \mathrm{e}^{6t}\cdot\Big[\frac{t}{2} - \frac{1}{2\cdot 6}\Big].$$

For α an arbitrary number, then,

17e†: $$z_\alpha(t) = \alpha\mathrm{e}^{-6t} + \big[\tfrac{t}{2} - \tfrac{1}{12}\big].$$ Hence

19c†: $$y_\alpha(t) = \Big[\alpha\mathrm{e}^{-6t} + \tfrac{t}{2} - \tfrac{1}{12}\Big]^{1/3}.$$

ZeroTar FOLDE. [This uses notation from the (17a) paragraph.] Because a "W" looks a bit like an upside-down "M", when FOLDE-ing I'll sometimes define

$$W(x) := \frac{1}{M(x)} \stackrel{\text{recall}}{=\!=} \mathrm{e}^{-B(x)}.$$

In this notation, soln (17e) is

$$y_\alpha(x) = \alpha\cdot W(x) + Q(x)\cdot W(x).$$

In particular, when target fnc G from (17a) is zero, our general soln reduces to $y_\alpha(x) = \alpha\cdot W(x)$. So if we just need *one* non-trivial soln, we can let $\alpha{=}1$, giving

$$y(x) = W(x) = 1\big/\mathrm{e}^{B(x)}.$$

The EXACT algorithm

[§2.4–**NSS9**. §2.4–**ZW8**.] Write your DE in form

20a: $$[\mathcal{N}(x,y)\cdot \frac{dy}{dx}] + \mathcal{M}(x,y) = 0.$$

Our goal is to describe y as an ***implicit solution***: We seek a non-trivial function $\mathbf{F}(\cdot,\cdot)$ so that each solution y to (20a) satisfies

20b: $$\mathbf{F}(x,y(x)) = \alpha,$$

for some constant α. [If we are interested in complex-valued solutions, then we will allow α to be a complex number.]

Step E1. Does

20c: $$\frac{\partial\mathcal{N}}{\partial x} = \frac{\partial\mathcal{M}}{\partial y} \ ?$$

If yes, then (20a) is **"an *exact DE*"**; this means [courtesy of our theorem] that there exists a differentiable fnc $\mathbf{F}(x,y)$ such that

20c′: $$\frac{\partial\mathbf{F}}{\partial y} = \mathcal{N} \quad\text{and}\quad \frac{\partial\mathbf{F}}{\partial x} = \mathcal{M}.$$

In this case, proceed to step (**E2**). Conversely, if (20a) is not exact, go to (**E1.1**) and (**E1.2**).

Step E2. Compute $\mathbf{F}()$ as follows. Compute two antiderivatives, and their difference:

$$\mathcal{B}(x,y) := \int^y \mathcal{N}(x,\widetilde{y})\,d\widetilde{y} \quad ;$$
$$\mathcal{A}(x,y) := \int^x \mathcal{M}(\widetilde{x},y)\,d\widetilde{x} \quad ;$$
$$\text{Diff}(x,y) := \mathcal{B}(x,y) - \mathcal{A}(x,y).$$

Since (20c′) holds, this difference $\text{Diff}(x,y)$ can be written as the difference between a pure function of y and a pure function of x. We do that next.

Step E3. Find functions $g(y)$ and $h(x)$ so that [this can usually be done by inspection]

20d: $$\text{Diff}(x,y) = g(y) - h(x).$$

[The pair of functions g,h is *almost* unique —adding a constant to g and the same constant to h, gives another a soln-pair.] One can compute a function $\mathbf{F}()$ which satisfies (20c′), by either

20e: $$\mathbf{F}(x,y) := \mathcal{A}(x,y) + g(y) \quad \textit{or}$$
$$\mathbf{F}(x,y) := \mathcal{B}(x,y) + h(x).$$

Step E4. Now use (20b) to discern what you need to know about $y(x)$, such as asymptotic behavior as $x \to \pm\infty$. You might do this by solving (20b) explicitly for $y(x)$, or you might use qualitative methods.

EXACT Example. U.F $y{=}y(x)$ is a soln to

20a∗: $$[8y + \sin(x)]y' + y\cos(x) - 3x^2 = 0.$$

With $\mathcal{N} := 8y + \sin(x)$ and $\mathcal{M} := y\cos(x) - 3x^2$, note

$$\mathcal{N}_x = 0 + \cos(x) = \cos(x) + 0 = \mathcal{M}_y;$$

happily (20a∗) is exact.

Anti-differentiating w.r.t y, then x, gives

$$\mathcal{B}(x,y) := \int^y \mathcal{N} \stackrel{\text{note}}{=\!=} 4y^2 + y\sin(x);$$
$$\mathcal{A}(x,y) := \int^x \mathcal{M} \stackrel{\text{note}}{=\!=} y\sin(x) - x^3. \quad \textit{Thus}$$
$$\mathcal{B} - \mathcal{A} = 4y^2 + x^3 = g(y) - h(x), \quad \textit{where}$$

we can define $g(y) := 4y^2$ and $h(x) := -x^3$. Hence

$$\mathbf{F}(x,y) := \mathcal{B}(x,y) + h(x)$$
$$= 4y^2 + y\sin(x) - x^3 \stackrel{\text{note}}{=\!=} \mathcal{A}(x,y) + g(y).$$

Consequently, each soln $y()$ to (20a∗), satisfies

$$4[y(x)]^2 + [y(x)\cdot\sin(x)] - x^3 = \alpha$$

for *some* number α.

Step E1.1. [§2.5–**NSS9**. §2.4–**ZW8**.] When (20a) is *not* exact, check to see if we can create an exact-ifying fnc $W(x)$, as follows. Compute

$$\text{20f:}\qquad C(x,y) := \frac{\mathcal{N}_x(x,y) - \mathcal{M}_y(x,y)}{\mathcal{N}(x,y)}.$$

Simplify $C(x,y)$ to see if it is a fnc of x only. If "no", then (20a) cannot be made exact by multiplying by a pure fnc of x. Try (**E1.2**), later in these notes.

If "yes", then write $C(x) := C(x,y)$. An exact-ifying factor $W(x)$ is a soln to DE

$$*:\qquad W'(x) + C(x)W(x) = 0.$$

Applying FOLDE, define $B() := \int C()$. Then

$$\text{20g:}\qquad W(x) := 1/\mathrm{e}^{B(x)}$$

satisfies (∗).

Finally, define two new functions

$$\text{20h:}\qquad \begin{aligned}\widehat{\mathcal{N}}(x,y) &:= \mathcal{N}(x,y)\cdot W(x) \qquad \text{and}\\ \widehat{\mathcal{M}}(x,y) &:= \mathcal{M}(x,y)\cdot W(x).\end{aligned}$$

Automatically, differential eqn

$$20a.1\text{:}\qquad \Big[\widehat{\mathcal{N}}(x,y)\cdot \tfrac{\mathrm{d}y}{\mathrm{d}x}\Big] + \widehat{\mathcal{M}}(x,y) = 0.$$

is exact. Apply steps (**E2**,**E3**,**E4**) to (20*a*.1).

EXACT Example of (E1.1)

Consider DE

$$\text{20a}\dagger\text{:}\qquad \underbrace{[x+1]\cdot 2y}_{\mathcal{N}(x,y)}\cdot y' + \underbrace{3\cdot[5+y^2]}_{\mathcal{M}(x,y)} = 0.$$

Is this Exact? Applying (**E1**), note

$$\text{20c}\dagger\text{:}\qquad \mathcal{N}_x - \mathcal{M}_y = 2y - 6y \stackrel{\text{note}}{=\!=} \text{-}4y$$

is *not* the zero-fnc, so (20a†) is not an exact DE. To attempt an exact-ifying factor, (**E1.5**), we compute

$$\text{20f}\dagger\text{:}\qquad C(x,y) := \frac{\text{-}4y}{[x+1]\cdot 2y} = \text{-}2/[x+1].$$

This is a pure fnc of x, so we anti-diff w.r.t x and get $B(x) := \text{-}2\cdot\log(x+1)$. Our exact-ifying factor is thus

$$W(x) := \mathrm{e}^{\text{-}B(x)} \stackrel{\text{note}}{=\!=} [x+1]^2.$$

Good! We now have Exact DE (20*a*.1), where

$$\text{20h}\dagger\text{:}\qquad \begin{aligned}\widehat{\mathcal{N}}(x,y) &= [x+1]^3\cdot 2y \qquad \textit{and}\\ \widehat{\mathcal{M}}(x,y) &:= 3\cdot[x+1]^2\cdot[5+y^2].\end{aligned}$$

***Applying* (E2), *then* (E3).** Anti-differentiating w.r.t y, respectively, x gives

$$\begin{aligned}\mathcal{B}(x,y) &:= \int^y \widehat{\mathcal{N}} \stackrel{\text{note}}{=\!=} [x+1]^3\cdot y^2;\\ \mathcal{A}(x,y) &:= \int^x \widehat{\mathcal{M}} \stackrel{\text{note}}{=\!=} [x+1]^3\cdot[5+y^2]. \text{ Thus}\\ \mathcal{B} - \mathcal{A} &\stackrel{\text{note}}{=\!=} \text{-}5\cdot[x+1]^3 = g(y) - h(x), \text{ where}\end{aligned}$$

we can define $g(y) := 0$ and $h(x) := 5\cdot[x+1]^3$. Finally, (20e) tells us that $\mathbf{F} = \mathcal{A} + g \stackrel{\text{note}}{=\!=} \mathcal{A}$.

Checking. Consider a fnc $y = y(x)$ satisfying

$$**\text{:}\qquad \text{Const} = [x+1]^3\cdot[5+y(x)^2].$$

Applying $\frac{\mathrm{d}}{\mathrm{d}x}$ hands us

$$0 = 3[x+1]^2\cdot[5+y(x)^2] + [x+1]^3\cdot 2y(x)\cdot y'(x).$$

Dividing by $[x+1]^2$ yields (20a†). *Nice. . .*

Step E1.2. When step (**E1.1**) fails, check for an exact-ifying fnc $W(y)$, as follows. Compute

$$\text{20i:} \qquad C(x,y) \;:=\; \frac{\mathcal{M}_y(x,y) - \mathcal{N}_x(x,y)}{\mathcal{M}(x,y)}\,.$$

Simplify $C(x,y)$ to see if it is a fnc of y alone. If "yes", write $C(y) := C(x,y)$. This time, exact-ifying factor $W(y)$ satisfies DE

$$*: \qquad W'(y) + C(y)W(y) \;=\; 0\,.$$

Applying FOLDE, define $B() := \int C()$. Then

$$W(y) \;:=\; 1/\mathsf{e}^{B(y)}$$

fulfills $(*)$. Define two new functions

$$\text{20j:} \qquad \begin{aligned} \widehat{\mathcal{N}}(x,y) \;&:=\; \mathcal{N}(x,y)\cdot W(y) \qquad \text{and} \\ \widehat{\mathcal{M}}(x,y) \;&:=\; \mathcal{M}(x,y)\cdot W(y)\,. \end{aligned}$$

Apply steps (**E2**,**E3**,**E4**) to DE

$$\text{20}a\text{.2:} \qquad [\widehat{\mathcal{N}}(x,y)\cdot \tfrac{\mathrm{d}y}{\mathrm{d}x}] + \widehat{\mathcal{M}}(x,y) \;=\; 0\,,$$

which is exact.

EXACT Example of (E1.2)

Consider $y{=}y(x)$ in

$$\text{20a‡:} \qquad \underbrace{x^2}_{\mathcal{N}(x,y)}\cdot y' \;-\; \underbrace{[y^2 + 2xy]}_{\mathcal{M}(x,y)} \;=\; 0\,.$$

Firstly,

$$\mathcal{N}_x - \mathcal{M}_y \;=\; 2x - \big[\text{-}[2y+2x]\big] \overset{\text{note}}{=\!=} 2[y+2x]$$

is not the zero-fnc, so (20a‡) is not exact. Secondly, ratio

$$\frac{\mathcal{N}_x - \mathcal{M}_y}{\mathcal{N}} \;=\; \frac{2\cdot[y+2x]}{x^2}$$

is not a pure fnc of x, so (**E1.1**) is inapplicable.

Applying (**E1.2**), we compute $C(x,y)$ as

$$\text{20i‡:} \qquad \frac{\mathcal{M}_y - \mathcal{N}_x}{\mathcal{M}} \overset{\text{note}}{=\!=} \frac{\text{-}[2\cdot[y+2x]]}{\text{-}[y^2 + 2xy]} \overset{\text{note}}{=\!=} \frac{2}{y}\,.$$

Yes! –this <u>is</u> a pure fnc of y. Applying FOLDE, we anti-diff w.r.t y, obtaining $B(y) := 2\cdot\log(y)$. Our exact-ifying factor is thus

$$W(y) \overset{\text{def}}{=\!=} \mathsf{e}^{\text{-}B(y)} \overset{\text{note}}{=\!=} 1/y^2\,.$$

Multiplying (20a‡) by $\frac{1}{y^2}$ gives exact $\widehat{\mathcal{N}}\cdot y' + \widehat{\mathcal{M}} = 0$, where

$$\text{20j‡:} \qquad \begin{aligned} \widehat{\mathcal{N}}(x,y) \;&:=\; \frac{x^2}{y^2} \qquad \text{and} \\ \widehat{\mathcal{M}}(x,y) \;&:=\; \text{-}\Big[1 + \frac{2x}{y}\Big]\,. \end{aligned}$$

Applying **(E2,E3).** Anti-differentiating w.r.t y and x, yields

$$\begin{aligned} \mathcal{B}(x,y) \;&:=\; \int^y \widehat{\mathcal{N}} \overset{\text{note}}{=\!=} -\frac{x^2}{y}\,; \\ \mathcal{A}(x,y) \;&:=\; \int^x \widehat{\mathcal{M}} \overset{\text{note}}{=\!=} \text{-}\Big[x + \frac{x^2}{y}\Big]\,. \text{ Thus} \\ \mathcal{B} - \mathcal{A} \;&=\; x \;=\; g(y) - h(x)\,, \text{ where} \end{aligned}$$

we define $g(y) := 0$ and $h(x) := \text{-}x$. Finally, (20e) tells us that $\mathbf{F} = \mathcal{A} + g \overset{\text{note}}{=\!=} \mathcal{A}$.

Checking. Consider a fnc $y{=}y(x)$ satisfying

$$**: \qquad \alpha \;=\; \text{-}\Big[x + \frac{x^2}{y(x)}\Big]\,,$$

for some number α. Applying $\frac{\mathrm{d}}{\mathrm{d}x}$ produces that

$$0 \;=\; \text{-}\Big[1 + \frac{2xy - x^2y'}{y^2}\Big] \overset{\text{note}}{=\!=} \frac{x^2y' - 2xy - y^2}{y^2}\,.$$

Multiplying by y^2 yields (20a‡), as desired.

In this instance, we can actually solve $(**)$ for $y()$ as

$$y_\alpha(x) \;=\; \frac{\text{-}x^2}{\alpha + x}\,.$$

Nifty...

Exactifying-factor theory

For fncs $\mathcal{N}{=}\mathcal{N}(x,y)$ and $\mathcal{M}{=}\mathcal{M}(x,y)$, suppose DE $\boxed{\mathcal{N}y' + \mathcal{M} = 0}$ is *not* exact. What property would a fnc $W{=}W(x,y)$ have to possess in order that DE

$$\dagger: \quad [\mathcal{N}W]y' + [\mathcal{M}W] = 0$$

be exact? Exactness requires equality of

$$*: \quad \begin{aligned} [\mathcal{N}\cdot W]_x &\stackrel{\text{note}}{=\!=} \mathcal{N}W_x + \mathcal{N}_x W \qquad \textit{with} \\ [\mathcal{M}\cdot W]_y &\stackrel{\text{note}}{=\!=} \mathcal{M}W_y + \mathcal{M}_y W\,. \end{aligned}$$

That is, (†) is exact IFF

$$\ddagger: \quad 0 = \mathcal{N}W_x - \mathcal{M}W_y + [\mathcal{N}_x - \mathcal{M}_y]W\,.$$

Alas, PDE (‡) is likely as difficult as the original DE.

IDEA: Could a pure fnc of x be an exactifying-factor? If $W{=}W(x)$, then W_y is zero, so (‡) becomes

$$\ddagger_x: \quad 0 = W_x + \tfrac{\mathcal{N}_x - \mathcal{M}_y}{\mathcal{N}} W\,.$$

This effectively[♡5] forces ratio $\frac{\mathcal{N}_x - \mathcal{M}_y}{\mathcal{N}}$ to be x-pure. Hence ($\ddagger_x$) is a FOLDE [the easy =0 case]. This explains where coeff-fnc (20f) came from. Similarly, were W a pure fnc of y, then (‡) reduces to

$$\ddagger_y: \quad 0 = W_y + \tfrac{\mathcal{M}_y - \mathcal{N}_x}{\mathcal{M}} W\,,$$

explaining coeff-fnc (20i).

2-variable exactifying-factor. Verify that pair

$$\text{20k.1:} \quad \begin{aligned} \mathcal{N} &= \mathcal{N}(x,y) := 5xy^2 + 3x^2 \quad \textit{and} \\ \mathcal{M} &= \mathcal{M}(x,y) := 2y^3 + 3xy \end{aligned}$$

is *not* an exact-pair. Show that $\boxed{H = H(x,y) := xy^2}$ *is* an exactifying-factor for the **(**$\mathcal{N},\mathcal{M}$**)** pair.

Soln to 2-V E-F. Firstly, derivatives

$$\begin{aligned} \mathcal{N}_x &\stackrel{\text{note}}{=\!=} 5y^2 + 3\cdot 2x \qquad \textit{and} \\ \mathcal{M}_y &\stackrel{\text{note}}{=\!=} 2\cdot 3y^2 + 3x \end{aligned}$$

are *not* equal, showing pair **(**$\mathcal{N},\mathcal{M}$**)** not exact.

Define products

$$\begin{aligned} \widehat{\mathcal{N}} &:= \mathcal{N}H \stackrel{\text{note}}{=\!=} 5x^2y^4 + 3x^3y^2 \qquad \textit{and} \\ \widehat{\mathcal{M}} &:= \mathcal{M}H \stackrel{\text{note}}{=\!=} 2xy^5 + 3x^2y^3\,. \end{aligned}$$

Observe that *these* derivatives,

$$\begin{aligned} [\widehat{\mathcal{N}}]_x &\stackrel{\text{note}}{=\!=} 5\cdot 2xy^4 + 3\cdot 3x^2y^2 \qquad \textit{and} \\ [\widehat{\mathcal{M}}]_y &\stackrel{\text{note}}{=\!=} 2\cdot 5xy^4 + 3\cdot 3x^2y^2\,, \end{aligned}$$

are indeed equal.

In the spirit of IAATYD**M**TWIAYTD, applying the EXACT algorithm produces fnc

$$\mathbf{F}(x,y) := x^2y^5 + x^3y^3$$

s.t $\mathbf{F}_y = \widehat{\mathcal{N}}$ and $\mathbf{F}_x = \widehat{\mathcal{M}}$. In consequence, each [complex] number α gives implicit soln

$$\mathbf{F}\big(x, y(x)\big) = \alpha$$

to DE

$$\mathcal{N}\big(x, y(x)\big)\cdot y'(x) + \mathcal{M}\big(x, y(x)\big) = 0\,,$$

for the $\mathcal{N}$ and $\mathcal{M}$ defined in (20k.1). ♦

[♡5] *Weasel word alert!* I'll explain in class.

Logistic model [§3.2–NSS9, P.98]

Suppose $p = p(t)$ measures the size of a population at time t. Let ⓟ be a placeholder for the units of p. [If $p(t)$ measures the weight of bacteria in a petri dish at time t, then ⓟ might mean ounces. If $p(t)$ is a count of individuals then ⓟ indicates no units.] Suppose the population has ***natural birth-multiplier*** $\mathbf{B}>0$, in units 1/ⓣ. [Agree that $\mathbf{B} :: \frac{1}{ⓣ}$ means that $\mathbf{B}$ is in abstract units 1/ⓣ.] Were there no constraints, the DE[♡6] would be

$$p' = \mathbf{B}\cdot p, \quad \textit{with soln} \quad p(t) = \mathbf{p}_0 \cdot \mathrm{e}^{\mathbf{B}\cdot t}.$$

A more realistic model has a ***carrying capacity*** $\mathbf{C}>0$ [with $\mathbf{C}$ :: ⓟ], which is the maximum population that the environment can sustain. As long as $0<p()<\mathbf{C}$, the population continues to grow, albeit more and more slowly. When $p>\mathbf{C}$, then the population declines [deaths exceed births], asymptotically approaching $\mathbf{C}$. The form of the DE might be

$$\text{12a:} \qquad \frac{\mathrm{d}p}{\mathrm{d}t}(t) = \Big[\mathbf{B}\cdot F\big(p(t)\big)\Big]\cdot p(t),$$

where $\mathbf{B}\cdot F\big(p(t)\big)$ is the birth-mult @t. This $F()$ has $\lim_{p\to\mathbf{C}} F(p) = 0$, $\lim_{p\searrow 0} F(p) = 1$, and likely should be continuous, and strictly decreasing, for $0<p<\mathbf{C}$.

The simplest such F is $F(p) := 1 - \frac{p}{\mathbf{C}}$. This engenders the *Logistic model* DE[♡7]

$$\text{12b:} \qquad \frac{\mathrm{d}p}{\mathrm{d}t} = \mathbf{B}\cdot[1-\tfrac{p}{\mathbf{C}}]\cdot p.$$

Solving (12b). Define $q(t) := \frac{p(t)}{\mathbf{C}}$. Thus

$$q' = \tfrac{1}{\mathbf{C}}p' = \mathbf{B}\cdot[1-q]\cdot\tfrac{1}{\mathbf{C}}\cdot p = \mathbf{B}\cdot[1-q]\cdot q. \quad \textit{I.e,}$$

$$\text{12c:} \qquad \frac{\mathrm{d}q}{\mathrm{d}t} = \mathbf{B}\cdot[1-q]\cdot q.$$

This DE separates[♡8] as $\frac{1}{q\cdot[1-q]}\,\mathrm{d}q = \mathbf{B}\,\mathrm{d}t$. Antidiffing RhS gives $\mathbf{B}t$. [Exer: DE (12c) is autonomous and 1st-order, so we don't need a CoI. Why?] Partial-fractioning gives

$$*: \qquad \frac{1}{q\cdot[1-q]} = \frac{1}{q} + \frac{1}{1-q}.$$

When $0<q<1$**.** Expression $(*)$ antidifferentiates to $\big[\log(q) - \log(1-q)\big]$. Exponentiating gives

$$\dagger: \qquad \mathrm{e}^{\mathbf{B}t} = \frac{q}{1-q} \overset{\text{note}}{=\!=} \frac{1}{1-q} - 1.$$

A soupçon of algebra yields

$$q = \frac{\mathrm{e}^{\mathbf{B}t}}{\mathrm{e}^{\mathbf{B}t}+1} = \frac{1}{1+\mathrm{e}^{-\mathbf{B}t}}.$$

Un-substituting, and using autonomy, hands us

$$\text{12d:} \qquad p(t) = \frac{\mathbf{C}}{1 + \mathrm{e}^{-\mathbf{B}\cdot[t-\boldsymbol{\tau}_{\text{Half}}]}},$$

where $p(\boldsymbol{\tau}_{\text{Half}})$ is half of $\mathbf{C}$.

Otherwise. If $q>1$ then $(*)$ antidifferentiates to $\big[\log(q) - \log(q-1)\big]$. Exponentiating produces $\frac{q}{q-1}$.

OTOHand, if $q<0$ then $(*)$ antidifferentiates to $\big[\log(-q) - \log(1-q)\big]$. Exponentiating results in $\frac{-q}{1-q} \overset{\text{note}}{=\!=} \frac{q}{q-1}$. Hence *both* $q>1$ and $q<0$ produce

$$\ddagger: \qquad \mathrm{e}^{\mathbf{B}t} = \frac{q}{q-1}.$$

Routine algebra cheerfully delivers

$$\text{12e:} \qquad p(t) = \frac{\mathbf{C}}{1 - \mathrm{e}^{-\mathbf{B}\cdot[t-\boldsymbol{\tau}_{\text{Asymp}}]}},$$

where this $p()$ has a vertical-asymptote at $t=\boldsymbol{\tau}_{\text{Asymp}}$.

Algebra. Both (12d,12e) rewrite as $p(t) = \frac{\mathbf{C}}{1+M\cdot\mathrm{e}^{-\mathbf{B}\cdot t}}$, where M

$$\overset{\text{by (12d)}}{=\!=} \mathrm{e}^{\mathbf{B}\cdot\boldsymbol{\tau}_{\text{Half}}}, \qquad \overset{\text{by (12e)}}{=\!=} -\mathrm{e}^{\mathbf{B}\cdot\boldsymbol{\tau}_{\text{Asymp}}},$$

respectively. Plugging t=0min into (12d,12e) says $\mathbf{p}_0$

$$\overset{\text{by (12d)}}{=\!=} \frac{\mathbf{C}}{1+\mathrm{e}^{\mathbf{B}\cdot\boldsymbol{\tau}_{\text{Half}}}}, \qquad \overset{\text{by (12e)}}{=\!=} \frac{\mathbf{C}}{1-\mathrm{e}^{-\mathbf{B}\cdot\boldsymbol{\tau}_{\text{Asymp}}}}$$

respectively. In both cases, then, $M = \frac{\mathbf{C}}{\mathbf{p}_0} - 1$. □

Unifying. The above algebra yielded a uniform description of (12d), (12e) and the $p() \equiv \mathbf{C}$ forward-stable equilibrium soln, as

[♡6] Sometimes called the ***Malthusian model*** because of ideas in **An Essay on the Principle of Population**, 1798, by *Thomas Robert Malthus*. However, I am unaware of evidence that Malthus wrote down a differential-eqn.

[♡7] Usually attributed to *Pierre-François Verhulst* in 1838.

[♡8] Dividing by $q\cdot[1-q]$ loses solns $q\equiv 0$ and $q\equiv 1$; i.e, loses $p\equiv 0$ⓟ and $p\equiv\mathbf{C}$. We'll regain these two equilibrium solns later.

12f: $$p(t) \;=\; \frac{\mathbf{C}}{1 + [\frac{\mathbf{C}}{\mathbf{p}_0} - 1] \cdot \mathsf{e}^{-\mathbf{B} \cdot t}}\,,$$

where $\mathbf{p}_0$ denotes the population at time 0.

Multiplying top&bottom by $\mathbf{p}_0$ unifies with the forward-unstable $p() \equiv 0$ equilibrium soln, giving

12g: $$p(t) \;=\; \frac{\mathbf{C} \cdot \mathbf{p}_0}{\mathbf{p}_0 \;+\; [\mathbf{C} - \mathbf{p}_0] \cdot \mathsf{e}^{-\mathbf{B}t}} \;=\; \frac{\mathbf{C} \cdot \mathbf{p}_0 \cdot \mathsf{e}^{\mathbf{B}t}}{\mathbf{C} \;+\; \mathbf{p}_0 \cdot [\mathsf{e}^{\mathbf{B}t} - 1]}\,.$$

Although derived in $\mathbb{R}$, please check, for all complex numbers $\mathbf{p}_0$,$\mathbf{B}$,$\mathbf{C}$ with $\mathbf{C} \neq 0$, that (12f,12g) satisfy DE (12b) for <u>all</u>[♡9] *complex* times t.

Cool stuff. . . .

Exer: Doubling-time. *Haffoweria* bacteria have an unconstrained (Malthusian model) ***doubling time***[♡10] of 30min. Compute the birth-multiplier, $\mathbf{B}$, for *Haffoweria*.

Soln. Define $\boldsymbol{\tau}_{\mathrm{Dbl}} := 30\mathsf{min}$. [It is often, but not always, good to give conceptual names to values]

The Malthusian model gives $p(t) \;=\; \mathbf{p}_0 \cdot \mathsf{e}^{\mathbf{B}t}$. So

$$2 \;=\; \frac{p(\boldsymbol{\tau}_{\mathrm{Dbl}})}{p(0\,\mathsf{min})} \;=\; \frac{\exp(\mathbf{B} \cdot \boldsymbol{\tau}_{\mathrm{Dbl}})}{1}\,.$$

Logarithmizing gives

$$\mathbf{B} \;=\; \frac{\log(2)}{\boldsymbol{\tau}_{\mathrm{Dbl}}} \;=\; \frac{\log(2)}{30\,\mathsf{min}} \;\approx\; 0.023\,\frac{1}{\mathsf{min}}\,. \quad \blacklozenge$$

[♡9] Well –. . . *essentially.* If $\mathbf{B} = \frac{0}{\mathsf{min}}$ or $\mathbf{C} = \mathbf{p}_0$, then the soln is constant. When $\mathbf{B} \neq \frac{0}{\mathsf{min}}$ *and* $\mathbf{C} \neq \mathbf{p}_0$, then the soln has a single (complex) time, $\boldsymbol{\tau}_{\mathrm{Asymp}}$, when the (12f)-denominator is zero.

[♡10] Apparently, *doubling time* is also called *generation time*.

Exer: Population-sampling. Reproduction of the fascinating *DiffTheorya* bacteria closely follows the logistic model (12b). Sharon Scientist designs a protocol to estimate $\mathbf{B}$ and $\mathbf{C}$ for *DiffTheorya*:

She put initial population $\mathbf{p}_0 = p(0\mathsf{min})$, into a petri dish, then measured the pop. at two later times $\mathbf{t}_1 < \mathbf{t}_2$. Prior to this, she used DiffyQ to derive the simplest time-ratio $\boldsymbol{\rho} := \frac{\mathbf{t}_2}{\mathbf{t}_1}$ for her protocol. *What time-ratio $\boldsymbol{\rho} = \frac{\mathbf{t}_2}{\mathbf{t}_1}$ [not nec. an integer] did Dr. Scientist use?*

Pop-samp, Theory. Formula (12f),

$$*: \qquad p(t) = \frac{\mathbf{C}}{1 + [\frac{\mathbf{C}}{\mathbf{p}_0} - 1] \cdot \mathsf{e}^{-\mathbf{B}\cdot t}},$$

suggests studying ratio

$$\frac{p(0\mathsf{min})}{p(t)} \stackrel{\text{note}}{=\!=} \frac{1 + M\mathsf{e}^{-\mathbf{B}\cdot t}}{1 + M}, \quad \text{where } M := \frac{\mathbf{C}}{\mathbf{p}_0} - 1.$$

Can we isolate $\mathbf{B}$? Observe that

$$H(t) := \frac{p(0\,\mathsf{min})}{p(t)} - 1 \stackrel{\text{note}}{=\!=} [\mathsf{e}^{-\mathbf{B}\cdot t} - 1] \cdot \frac{M}{1 + M}.$$

Define [we are not dividing by zero, as $\mathbf{t}_1 \neq 0\mathsf{min}$] ratio

$$\mathbf{R} := \frac{H(\mathbf{t}_2)}{H(\mathbf{t}_1)} = \frac{\mathsf{e}^{-\mathbf{B}\cdot\mathbf{t}_2} - 1}{\mathsf{e}^{-\mathbf{B}\cdot\mathbf{t}_1} - 1}.$$

With $\mathcal{N} := \mathsf{e}^{-\mathbf{B}\cdot\mathbf{t}_1}$ [for $\mathcal{N}$egative-expon], note $\mathcal{N}^{\boldsymbol{\rho}} = \mathsf{e}^{-\mathbf{B}\cdot\mathbf{t}_2}$. Thus

$$\mathbf{R} = \frac{\mathcal{N}^{\boldsymbol{\rho}} - 1}{\mathcal{N} - 1}.$$

When $\boldsymbol{\rho}$ is a posint, then,

$$\mathbf{R} = \mathcal{N}^{\boldsymbol{\rho}-1} + \mathcal{N}^{\boldsymbol{\rho}-2} + \cdots + \mathcal{N} + 1.$$

So the simplest useful ratio is $\boldsymbol{\rho} := \frac{\mathbf{t}_2}{\mathbf{t}_1} = 2$, whence $\boxed{\mathbf{R} = \mathcal{N} + 1}$. [Exer: What is wrong with using $\boldsymbol{\rho}=1$?]

BirthMult. Recall $\mathbf{R} \stackrel{\text{def}}{=\!=} \frac{[\mathbf{p}_0/\mathbf{p}_2] - 1}{[\mathbf{p}_0/\mathbf{p}_1] - 1}$. Hence

$$\mathcal{N} = \mathbf{R} - 1 = \frac{\frac{\mathbf{p}_0}{\mathbf{p}_2} - \frac{\mathbf{p}_0}{\mathbf{p}_1}}{\frac{\mathbf{p}_0}{\mathbf{p}_1} - 1} \stackrel{\times\frac{\mathbf{p}_1\mathbf{p}_2}{\mathbf{p}_1\mathbf{p}_2}}{=\!=\!=} \frac{\mathbf{p}_1\mathbf{p}_0 - \mathbf{p}_2\mathbf{p}_0}{\mathbf{p}_0\mathbf{p}_2 - \mathbf{p}_1\mathbf{p}_2}.$$

It's more convenient to work with $\mathcal{E} := \frac{1}{\mathcal{N}} \stackrel{\text{note}}{=\!=} \mathsf{e}^{\mathbf{B}\cdot\mathbf{t}_1}$, the rEciprocal, whence

$$\mathcal{E} = \frac{1}{\mathbf{R} - 1} = \left[\frac{\mathbf{p}_1 - \mathbf{p}_0}{\mathbf{p}_2 - \mathbf{p}_1}\right] \cdot \frac{\mathbf{p}_2}{\mathbf{p}_0} \stackrel{\text{note}}{\geq} 0. \quad \textit{Thus,}$$

12h:

$$\mathbf{B} = \frac{\log(\mathcal{E})}{\mathbf{t}_1} = \frac{1}{\mathbf{t}_1} \cdot \log\left(\left[\frac{\mathbf{p}_1 - \mathbf{p}_0}{\mathbf{p}_2 - \mathbf{p}_1}\right] \cdot \frac{\mathbf{p}_2}{\mathbf{p}_0}\right).$$

CarryingCapacity. Recall that $\mathcal{E} = \mathsf{e}^{\mathbf{B}\cdot\mathbf{t}_1}$. Plugging $\mathbf{t}_1$ in formula (12f) gives $\mathbf{p}_1 = \frac{\mathbf{C}}{1 + [\frac{\mathbf{C}}{\mathbf{p}_0} - 1]/\mathcal{E}}$. Solving for $\mathbf{C}$ delivers

$$\mathbf{C} = \mathbf{p}_1 \cdot \overbrace{\frac{\mathcal{E} - 1}{\mathcal{E} - \frac{\mathbf{p}_1}{\mathbf{p}_0}}}^{=: S}. \quad \textit{Algebra gives}$$

12i:

$$S = \frac{\mathbf{p}_0\mathbf{p}_1 + \mathbf{p}_1\mathbf{p}_2 - 2\mathbf{p}_0\mathbf{p}_2}{\mathbf{p}_1\mathbf{p}_1 - \mathbf{p}_0\mathbf{p}_2}.$$

Our $\mathcal{E}$ and S are scale-inv fncs of $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2$. ♦

Pop-samp computation. Dr. Sharon used $\mathbf{t}_1 = 13\mathsf{min}$ and $\mathbf{p}_0 := 2\,\mathsf{oz}$, measuring $\mathbf{p}_1 := p(13\mathsf{min}) = 5.792\,\mathsf{oz}$, and $\mathbf{p}_2 := p(26\mathsf{min}) = 11.987\,\mathsf{oz}$. Formulas (12h,12i) and floating-point arithmetic gave her

$$\dagger: \quad \mathbf{B} \approx 0.099999994 \frac{1}{\mathsf{min}} \quad \textit{and} \quad \mathbf{C} \approx 20.000008\mathsf{oz}.$$

Not bad, as I had employed formula-(∗) with

$$\mathbf{B} := \frac{1}{10\,\mathsf{min}} \quad \textit{and} \quad \mathbf{C} := 20\,\mathsf{oz}.$$

As a responsible researcher, Dr. S. repeats her experiment, this time *exceeding* the estimated Carrying-Cap, initializing $\mathbf{p}_0 := 50\,\mathsf{oz}$.

She measures $\mathbf{p}_1 := p(8\,\mathsf{min}) = 27.382\,\mathsf{oz}$, then later $\mathbf{p}_2 := p(16\,\mathsf{min}) = 22.756\,\mathsf{oz}$. [The pop. is dying off.]

Trusty dusty floating-point produces

$$\ddagger: \quad \mathbf{B} \approx 0.09999992 \frac{1}{\mathsf{min}} \quad \textit{and} \quad \mathbf{C} \approx 19.999996\mathsf{oz},$$

which is consistent with (†). ♦

12j: *Questions.* For *Diff Theory a* with $\mathbf{p}_0 := 50$ oz:
LQ1: *When (in past time) is the vertical asymptote? How could we verify it experimentally?*

In the petri dish, Sharon observes that *Mysteria* bacteria stabilizes at 30 oz. Seeded with $\mathbf{p}_0 := 2$ oz, she records $\mathbf{p}_1 = 7.274$ oz just 10 min later.
LQ2: *What is the birth-mult for Mysteria? Started from 2 oz, how many minutes later is the dish at half CarryCap for Mysteria?* □

Hyperbolic trigonometric functions

The ***hyperbolic*** versions of cos and sin are written cosh [rhyming with "josh"] and sinh [pronounced "cinch"]. For z,α,β complex,

13a:
$$\cosh(z) := \frac{e^z + e^{-z}}{2} \stackrel{\text{note}}{=\!=} \cos(\mathbf{i}z)\,,$$
$$\sinh(z) := \frac{e^z - e^{-z}}{2} \stackrel{\text{note}}{=\!=} -\mathbf{i}\sin(\mathbf{i}z)\,,$$
$$\exp(\pm z) = \cosh(z) \pm \sinh(z)\,.$$

The corresponding facts about cos(),sin() give

13b: $\cosh(z+2\boldsymbol{\pi}\mathbf{i}) = \cosh(z)$, $\sinh(z+2\boldsymbol{\pi}\mathbf{i}) = \sinh(z)$; [*period* $2\boldsymbol{\pi i}$]

13c: $\cosh(z+\boldsymbol{\pi}\mathbf{i}) = -\cosh(z)$, $\sinh(z+\boldsymbol{\pi}\mathbf{i}) = -\sinh(z)$; [*anti-period* $\boldsymbol{\pi i}$]

13d: $\cosh(z + \mathbf{i}\frac{\pi}{2}) = \mathbf{i}\sinh(z)$; [*translation-scale*]

13e: $\cosh^2 - \sinh^2 = 1^2$. [Pythagoras]

13f:
$$\cosh(\alpha\pm\beta) = \cosh(\alpha)\cosh(\beta) \pm \sinh(\alpha)\sinh(\beta),$$
$$\sinh(\alpha\pm\beta) = \cosh(\alpha)\sinh(\beta) \pm \sinh(\alpha)\cosh(\beta).$$

All zeros of cosh & sinh are pure imaginary. Further,

13g: $\operatorname{Range}(\cosh) = \mathbb{C} = \operatorname{Range}(\sinh)$.

Easily,

13h:
$$\cosh' = \sinh, \qquad \sinh' = \cosh,$$
$$\cosh'' = \cosh, \qquad \sinh'' = \sinh\,.$$

Routinely, the Maclaurin series are

13i:
$$\cosh(z) = 1 + \tfrac{z^2}{2!} + \tfrac{z^4}{4!} + \ldots = \sum_{n=0}^{\infty} \frac{z^{2n}}{[2n]!}\,.$$
$$\sinh(z) = z + \tfrac{z^3}{3!} + \tfrac{z^5}{5!} + \ldots = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{[2n+1]!}\,.$$

Posting race: Translation? We know that sin() is a translate of cos(); i.e $\sin(z) = \cos(z - \frac{\pi}{2})$. *Dis(Prove): Function* sinh() *is a translate of* cosh(). *I.e,* $\exists\mathbf{T}\in\mathbb{C}$ so that $\sinh(z) = \cosh(z-\mathbf{T})$.

Inverse hyperbolic functions on $\mathbb{R}$. To build invertible fncs, we restrict domains so that the restrictions are 1-to-1. Define ***restricted cosh***, ResCosh, to be cosh restricted to the non-negative reals, and define ResSinh, ***restricted sinh,cosh***, to be sinh but only on the reals. I.e

$$\operatorname{ResCosh} := \cosh\downharpoonright_{[0,\infty)} \quad \textit{and} \quad \operatorname{ResSinh} := \sinh\downharpoonright_{\mathbb{R}}\,.$$

Easily, ResCosh and ResSinh are strictly increasing on their domains, indeed, are bijections

$$\operatorname{ResCosh}: [0,\infty) \hookrightarrow\!\!\!\!\rightarrow [1,\infty) \quad \textit{and} \quad \operatorname{ResSinh}: \mathbb{R} \hookrightarrow\!\!\!\!\rightarrow \mathbb{R}\,,$$

hence have inverse fncs

$$\operatorname{acosh} := \operatorname{ResCosh}^{-1} \quad \textit{and} \quad \operatorname{asinh} := \operatorname{ResSinh}^{-1}\,.$$

13j: **Hyperbolic inverses.** *Function* acosh() *bijects* $[1,\infty)$ *onto* $[0,\infty)$, *and* asinh $:\mathbb{R}\hookrightarrow\!\!\!\!\rightarrow\mathbb{R}$, *by*

†: $\operatorname{acosh}(t) = \log\Big(t + \sqrt{t^2-1}\Big)$, $\operatorname{acosh}'(t) = \frac{1}{\sqrt{t^2-1}}$,

‡: $\operatorname{asinh}(t) = \log\Big(t + \sqrt{t^2+1}\Big)$, $\operatorname{asinh}'(t) = \frac{1}{\sqrt{t^2+1}}$. ◊

Pf for acosh. Target $\mathbf{t}\in[1,\infty)$ asks for *the* $z\in[0,\infty)$ with $\cosh(z) = \mathbf{t}$. Set $E := e^z \stackrel{\text{note}}{\geq} e^0 = 1$. Expanding, $E + \frac{1}{E} = 2\mathbf{t}$. Thus $E^2 - 2\mathbf{t}E + 1 = 0$. Hence E is one of $[\mathbf{t}\pm\sqrt{\mathbf{t}^2-1}]$.

Were $E = [\mathbf{t}-\sqrt{\mathbf{t}^2-1}]$, then $\mathbf{t}-\sqrt{\mathbf{t}^2-1}\geq 1$, i.e, $\mathbf{t}-1\geq\sqrt{\mathbf{t}^2-1}$. Both sides are non-neg., so squaring preserves order, giving $\mathbf{t}^2+1-2\mathbf{t}\geq\mathbf{t}^2-1$. Thus $1\geq\mathbf{t}$; but that branch of square-root does not extend to $[1,\infty)$. So $E = \mathbf{t}+\sqrt{\mathbf{t}^2-1}$, whence LhS(†). ♦

Proof for asinh. For target $\mathbf{t}\in\mathbb{R}$ we seek *the* $z\in\mathbb{R}$ with $\sinh(z){=}\mathbf{t}$. With $E := e^z$, then, $E - \frac{1}{E} = 2\mathbf{t}$, so $E^2 - 2\mathbf{t}E - 1 = 0$. Thus $E\in[\mathbf{t}\pm\sqrt{\mathbf{t}^2+1}]$. But $[\mathbf{t}-\sqrt{\mathbf{t}^2+1}]$ is not >0. Hence $E = \mathbf{t}+\sqrt{\mathbf{t}^2+1}$, whence LhS(‡). ♦

Pf for asinh′. The Chain rule says $[f\circ g]' = [f'\circ g]\cdot g'$. With $f := \sinh$ and $g := \operatorname{asinh}$, for $t{\in}\mathbb{R}$ note

$$\begin{aligned}\sinh'(\operatorname{asinh}(t)) &= \cosh(\operatorname{asinh}(t)) \\ &\stackrel{**}{=\!=} \sqrt{\cosh^2(\operatorname{asinh}(t))} \\ &= \sqrt{\sinh^2(\operatorname{asinh}(t))+1} = \sqrt{t^2+1}\,.\end{aligned}$$

[Eqn (∗∗) holds, since sinh(t) is real, and cosh() is non-negative on $\mathbb{R}$.] Multiplying both sides by $\operatorname{asinh}'(t)$ produces

$$1 \stackrel{\text{note}}{=\!=} [\sinh\circ\operatorname{asinh}]'(t) = \sqrt{t^2+1}\cdot\operatorname{asinh}'(t)\,.$$

The proof for acosh′ is similar. ♦

Hyperbolic cosh , sinh solve certain classic DfyQs.

13k: Lemma. *For* $\boldsymbol{\alpha}$ *complex, fnc* $f(z) := \sinh(z - \boldsymbol{\alpha})$ *is a soln to*

$$[f']^2 \;=\; 1^2 \;+\; f^2.$$

The only other analytic solutions [*courtesy* FTODE] *are constant functions* $f() \equiv \pm\boldsymbol{i}$.

Integrating f *shows that the non-constant analytic solns to*

‡: $$[g'']^2 \;=\; 1^2 \;+\; [g']^2.$$

are $g(z) := \boldsymbol{\beta} + \cosh(z - \boldsymbol{\alpha})$, *for* $\boldsymbol{\beta}, \boldsymbol{\alpha} \in \mathbb{C}$. ♢

Proof. As $\sinh(z)^2 \;=\; \frac{1}{2^2}\Big[\mathsf{e}^{2z} + \mathsf{e}^{-2z} \;-\; 2\Big]$, so

$$\begin{aligned} 1^2 + [\sinh(z)]^2 \;&=\; \tfrac{1}{2^2}\Big[\mathsf{e}^{2z} + \mathsf{e}^{-2z} \;+\; 2\Big] \\ &=\; [\cosh(z)]^2 \stackrel{\text{note}}{=\!=} [\sinh'(z)]^2 \,. \end{aligned}$$ ♦

Derivation of hanging cable

Consider a hanging cable whose position is the graph of height fnc $y=h(x)$. As usual, use y' for $h'(x)$.

SETTINGS: A ***hanging cable*** (HC) only supports its own weight; the curve is called a ***catenary***. In the ***suspension bridge cable*** (SBC) setting, the cable supports the (horizontal) suspension-bridge ***deck***; we assume a massive deck compared to the cable-weight.

To normalize the notation, arrange the coordinate system so that *the lowest point of the cable is above* $x=0$ and hence y' is zero. Call this lowest point $(0, h(0))$ the ***vertex*** of the cable. We define three physical constants:

- T is the *tension* in the cable at its vertex. (The cable is horizontal here, so this is also its *horizontal* component of Tension.) This T has units Ⓦ.
- S is the *weight-per-distance* (ie, denSity) of the load at the cable's vertex. (So S is the limit as $x\searrow 0$ of $\frac{1}{x}$ times the weight on the cable-system above interval $[0, x]$.) This S has units Ⓦ/Ⓓ.
- R is the *ratio* $\mathsf{S}/\mathsf{T} :: \frac{1}{Ⓓ}$. Use $\mathsf{Q} := \frac{1}{\mathsf{R}} = \mathsf{T}/\mathsf{S} ::$ Ⓓ.

Cable tension. Let $\boldsymbol{\tau} = \boldsymbol{\tau}(x)$ denote the tension in the cable above x. Let $\boldsymbol{\tau}_{\text{Ver}}$ and $\boldsymbol{\tau}_{\text{Hor}}$ denote the vertical and horizontal components of tension; so $\boldsymbol{\tau}, \boldsymbol{\tau}_{\text{Ver}}, \boldsymbol{\tau}_{\text{Hor}}$ all have units Ⓦ.

Gravity acts only vertically. Were there points $x_0 < x_1$ with $\boldsymbol{\tau}_{\text{Hor}}(x_0) \neq \boldsymbol{\tau}_{\text{Hor}}(x_1)$, then the cable above interval $[x_0, x_1]$ would move horizontally. Since it does not, the fnc $\boldsymbol{\tau}_{\text{Hor}}()$ is a constant. So $\boxed{\boldsymbol{\tau}_{\text{Hor}} \equiv \mathsf{T}}$. Since ratio $\frac{\boldsymbol{\tau}_{\text{Ver}}}{\boldsymbol{\tau}_{\text{Hor}}}$ equals the cable slope y', necessarily

†: $$y'() = \frac{1}{\mathsf{T}} \cdot \boldsymbol{\tau}_{\text{Ver}}() .$$

Different values of T engender different cable shapes. [We'll discover that the suspension bridge cable is a parabola; different T-values produce different parabolae.]

Cable loading. Let $W(x)$ denote the *weight* of the cable above interval $[0, x]$. We will describe $W()$ as a product
$$W(x) = \mathsf{S} \cdot \Lambda(x)$$
so $\Lambda(x)$ has units Ⓓ. The meaning of $\Lambda(5\mathsf{ft})$ is the length which, were the cable-loading to have constant density S, would weigh the same as the cable-system above the interval $[0\mathsf{ft}, 5\mathsf{ft}]$.

Weight and tension. For $0 \le x_0 \le x_1$, the loading on the cable above an interval $[x_0, x_1]$ must equal the difference $\boldsymbol{\tau}_{\text{Ver}}(x_1) - \boldsymbol{\tau}_{\text{Ver}}(x_0)$ of the vertical components of tension. As $x=0$ is the lowest pt of the cable, necessarily $\boldsymbol{\tau}_{\text{Ver}}(0)$ is zero. For all $x \in \mathbb{R}$, then, $\boldsymbol{\tau}_{\text{Ver}}(x)$ equals $W(x)$. Hence
$$y'(x) \overset{\text{by (†)}}{=\!=} \frac{1}{\mathsf{T}} \cdot W(x) = \frac{1}{\mathsf{T}} \cdot \mathsf{S} \cdot \Lambda(x) ,$$
We rewrite this as

14: $$y'() = \mathsf{R} \cdot \Lambda() , \quad \textit{with} \\ y'(0) = 0, \quad \textit{and} \quad y(0) = 0 ,$$

where we tacked on initial conditions that the cable-vertex has horizontal tangent, and height zero.

This (14) is our IVP for cable problems with arbitary loading. We now solve it for two $\Lambda()$ load functions.

The Suspension Bridge solution

For the suspension bridge, $W(x) = \mathsf{S} \cdot x$. So $\Lambda(x) = x$. Integrating (14) thus produces height

15: $$h(x) = \tfrac{1}{2}\mathsf{R} \cdot x^2 = \tfrac{1}{2}\tfrac{\mathsf{S}}{\mathsf{T}}x^2 .$$

Note that the RhS has units $\frac{1}{ⓓ} \cdot ⓓ^2$. This equals ⓓ, which indeed is the abstract unit for height.

The Hanging Cable (catenary) solution

For the hanging cable, whose only load is itself,

$$W(x) := \mathsf{S} \cdot [\text{Cable arclength above } [0,x]] .$$

Consequently,

HC: $$\Lambda(x) = \int_0^x \sqrt{1^2 + h'(\tilde{x})^2}\, d\tilde{x} . \quad \textit{By FTC, then,}$$
$$\Lambda' = \sqrt{1 + [h']^2} .$$

Rather than compute the integral, we instead differentiate DE (14) to produce

$$h'' = \mathsf{R} \cdot \sqrt{1 + [h']^2} .$$

Squaring this gives

∗: $$[h'']^2 = \mathsf{R}^2 \cdot [1 + [h']^2] \overset{\text{same}}{=\!=} [1 + [h']^2]/\mathsf{Q}^2 .$$

Claim: $\boxed{h(x) := \frac{1}{\mathsf{R}}\cosh(\mathsf{R}x)}$ *satisfies* (∗).

Note $h'(x) = \cosh'(\mathsf{R}x)$. And $h''(x) = \mathsf{R}\cosh''(\mathsf{R}x)$. So

$$[h''(x)]^2 = \mathsf{R}^2 [\cosh''(\mathsf{R}x)]^2$$
$$\overset{\text{by (13k)}}{=\!=} \mathsf{R}^2 \Big[1 + [\cosh'(\mathsf{R}x)]^2\Big]$$
$$= \mathsf{R}^2 \Big[1 + [h'(x)]^2\Big] ,$$

as desired. Further, $h'(0) = \sinh(0) = 0$. Thus: *In the* HC *case, the soln to* (14) *is catenary* [*recall* $\mathsf{Q} = \frac{1}{\mathsf{R}}$]

16a: $$h(x) = \mathsf{Q} \cdot \Big[\cosh\Big(\frac{x}{\mathsf{Q}}\Big) - 1\Big].$$ *Or, letting vertex-height be non-zero,*
$$h(x) = \mathsf{Q} \cdot \cosh\Big(\frac{x}{\mathsf{Q}}\Big) = \frac{\mathsf{T}}{\mathsf{S}} \cdot \cosh\big(\tfrac{\mathsf{S}}{\mathsf{T}}\, x\big).$$

16b: Lemma. *The length of cable above interval* $[x_0, x_1]$ *is*

$$\mathrm{Len}(\textit{cable}) = \mathsf{Q} \cdot \Big[\sinh\Big(\frac{x_1}{\mathsf{Q}}\Big) - \sinh\Big(\frac{x_0}{\mathsf{Q}}\Big)\Big]. \quad \diamondsuit$$

Proof. Eqn (14) says our $\Lambda(x)$ equals

$$\mathsf{Q} \cdot h'(x) \overset{\text{by (16a)}}{=\!=} \mathsf{Q} \cdot \sinh\Big(\frac{x}{\mathsf{Q}}\Big) . \quad ♦$$

16c: Distance between poles? *An* 80ft *cable hangs between two* 50ft *poles, with lowest point* 20ft *above the ground. How far apart are the poles?* ♢

Prelim to (16c). We give symbolic names to the quantities. Let

$$L := \begin{bmatrix}\text{ArcLength from}\\ \text{vertex to a pole}\end{bmatrix} = \tfrac{1}{2}\cdot 80\text{ft} = 40\text{ft};$$

$$V := \begin{bmatrix}\text{Vertical dist. from}\\ \text{vertex to pole top}\end{bmatrix} = [50-20]\text{ft} = 30\text{ft};$$

$$\mathbf{z} := \begin{bmatrix}\text{HoriZontal distance}\\ \text{from vertex to pole}\end{bmatrix} = [\textit{Not yet known}].$$

Good eng. practice; LOWER/UPPER BNDS ON z:

£: $$L - V < z < \sqrt{L^2 - V^2}.$$

LOWER BND: The poles would be closer if the cable ran down the pole, then horizontally out to the vertex.

UPPER BND: The poles would be further apart if the cable ran straight from the pole-top to the vertex. This distance, says Pythagoras, $\sqrt{L^2 - V^2} = \sqrt{4^2 - 3^2}\cdot 10\text{ft} = \sqrt{7}\cdot 10\text{ft} \approx 26.457\text{ft}$.

We *expect* pole-separation, $2\mathbf{z}$, to satisfy

££: $$20\text{ft} < 2z < 53\text{ft}.$$

If our computation yields a value *not* in this range, we temporarily halt pole construction, and figure out WHAT WENT WRONG? WHERE?: The Four W's. □

Soln to (16c). We'll prove that the corresponding Q is

16c.1: $$\mathsf{Q} \overset{?}{=} \frac{L^2 - V^2}{2V} = \frac{35}{3}\text{ft}.$$

Lemma 16b applied with $x_1 := \mathbf{z}$ and $x_0 := 0\,\text{ft}$, gives

16c.2: $$\begin{aligned} L/\mathsf{Q} &= \sinh(\mathbf{z}/\mathsf{Q}). \quad \textit{Hence}\\ \operatorname{asinh}(L/\mathsf{Q}) &= \mathbf{z}/\mathsf{Q}. \quad \textit{So,}\\ \mathbf{z} &= \mathsf{Q}\cdot\operatorname{asinh}(L/\mathsf{Q}). \end{aligned}$$

As the question asks for $2\mathbf{z}$, our (16c.1) *would* give

‡: $$\begin{aligned} 2\mathbf{z} &= \frac{L^2 - V^2}{V}\cdot\operatorname{asinh}\Big(\frac{L\cdot 2V}{L^2 - V^2}\Big)\\ &\overset{\text{by (13j)}}{=\!=} \frac{L^2 - V^2}{V}\cdot\log\Big(\tfrac{L\cdot 2V}{L^2 - V^2} + \sqrt{[\tfrac{L\cdot 2V}{L^2 - V^2}]^2 + 1}\Big)\\ &= \tfrac{70}{3}\text{ft}\cdot\log\Big(\tfrac{24}{7} + \sqrt{[\tfrac{24}{7}]^2 + 1}\Big) \quad \begin{bmatrix}\text{NB: Pythag triple}\\ 7^2 + 24^2 = 25^2.\end{bmatrix}\\ &= \tfrac{70}{3}\cdot\log(7)\,\text{ft} \approx 45.4046\,\text{ft}. \end{aligned}$$

Proving (16c.1). Our (16a) and (16b) give, respectively,

$$\begin{aligned} \mathsf{R}V &= \cosh(\mathsf{R}\mathbf{z}) - \cosh(\mathsf{R}\cdot 0\text{ft}) = \cosh(\mathsf{R}\mathbf{z}) - 1,\\ \mathsf{R}L &= \sinh(\mathsf{R}\mathbf{z}) - \sinh(\mathsf{R}\cdot 0\text{ft}) = \sinh(\mathsf{R}\mathbf{z}). \end{aligned}$$

Thus $1 = [\cosh^2 - \sinh^2]$ equals

$$[\mathsf{R}V + 1]^2 - [\mathsf{R}L]^2 = [\mathsf{R}^2V^2 + 2\mathsf{R}V + 1] - \mathsf{R}^2L^2.$$

Subtracting 1 from both sides, then dividing by R, yields

$$2V = \mathsf{R}[L^2 - V^2].$$

Multiplying by $\frac{\mathsf{Q}}{2V}$ delivers (16c.1), as desired. ♦

16d: Same poles: Tension. *The previous cable has density* $\mathsf{S} := \frac{1}{5}\frac{\mathsf{lb}}{\mathsf{ft}}$. *What is the cable-tension at the vertex? What is the highest tension in the cable; where?* ♢

Soln to (16d). From our defn of Q, the vertex-tension is
$$\mathsf{T} \stackrel{\text{def}}{=} \boldsymbol{\tau}_{\text{Hor}}(0) \stackrel{\text{def}}{=} \mathsf{S}\cdot\mathsf{Q} \stackrel{\text{by (16c.1)}}{=\!=\!=} \frac{1}{5}\frac{\mathsf{lb}}{\mathsf{ft}}\cdot\frac{35}{3}\mathsf{ft} = \frac{7}{3}\,\mathsf{lb}.$$

Since $\boldsymbol{\tau}_{\text{Hor}}()$ is constant, the highest (in both senses!) tension is where the cable joins the pole; where $\boldsymbol{\tau}_{\text{Ver}}()$ is highest. That value is

16d.1: $$\boldsymbol{\tau}_{\text{Ver}}(\mathbf{z}) = \begin{bmatrix}\text{Cable weight from}\\ \text{vertex to pole}\end{bmatrix} = \mathsf{S}\cdot L = 8\,\mathsf{lb}.$$

Pythagoras tells us

∗: $$\boldsymbol{\tau}(x)^2 = \mathsf{T}^2 + [\boldsymbol{\tau}_{\text{Ver}}(x)]^2.$$

In particular, $\boldsymbol{\tau}(\mathbf{z})^2 = [\mathsf{SQ}]^2 + [\mathsf{S}L]^2$. Thus, max tension is

16d.2: $$\boldsymbol{\tau}(\mathbf{z}) = \mathsf{S}\cdot\sqrt{\mathsf{Q}^2 + L^2} = \mathsf{S}\cdot\frac{125}{3}\mathsf{ft} = \frac{25}{3}\mathsf{lb}.$$ ♦

Alt (16d). Note $\boldsymbol{\tau}_{\text{Ver}}(x) = \mathsf{T}\cdot h'(x) = \mathsf{T}\cdot\sinh(\frac{x}{\mathsf{Q}})$. So (∗) and identity $\mathsf{T}^2\cdot[1^2 + \sinh^2] = \mathsf{T}^2\cdot\cosh^2$ give

16d.3: $$\boldsymbol{\tau}(x) = \mathsf{T}\cdot\cosh(x/\mathsf{Q}).$$

Maximum tension is thus

16d.4: $$\boldsymbol{\tau}(\mathbf{z}) = \mathsf{T}\cdot\cosh(\mathbf{z}/\mathsf{Q}).$$

[Exer: % The righthand sides of (16d.2) and (16d.4) are equal.] ♦

16e: Breaking point. *On a planet with surface acceleration* $A := 10\frac{\mathsf{m}}{\mathsf{sec}^2}$, *an* 80m *long cable has mass* 16kg. *Its breaking tension is* 100 N. [A Newton is $\mathsf{N} = [\mathsf{kg}\cdot\mathsf{m}]/[\mathsf{sec}^2]$.] *What is the maximum span before this cable breaks?* ♢

Prelim to (16e). Looking at half the cable, from vertex to one pole:

$$L := \tfrac{1}{2}\cdot 80\mathsf{m} = 40\mathsf{m}, \quad \text{is the arcLength;}$$
$$\mathbf{z} := \begin{bmatrix}\text{HoriZontal distance}\\ \text{from vertex to pole}\end{bmatrix} = [\textit{Not yet known}];$$
$$W := [\text{Weight of half the cable}] = 8\,\mathsf{kg}\cdot A = 80\,\mathsf{N};$$
$$X := [\text{Maximum tension}] = 100\,\mathsf{N}.$$

The cable weight-density is $\mathsf{S} = W/L = 2\frac{\mathsf{N}}{\mathsf{m}}$.

Lower/Upper bnds are: $0\mathsf{m} < z < L = 40\mathsf{m}$. The 1st inequality is *strict*, since the length of cable hanging straight down needed to break the cable is $\frac{X}{\mathsf{S}} = \frac{100}{2}\mathsf{m} = 50\mathsf{m} \stackrel{\text{strict}}{>} L$. The 2nd inequality is also strict, since the breaking tension is *strictly* less than ∞. □

Soln (16e). We need $\mathbf{z}$ to satisfy $\boldsymbol{\tau}(\mathbf{z}) = X$. From (16d.2), then, $X^2 = \mathsf{S}^2\mathsf{Q}^2 + [\mathsf{S}L]^2$. And (16d.1) says $\mathsf{S}L = W$. Hence
$$\mathsf{Q} = \frac{1}{\mathsf{S}}\cdot\sqrt{X^2 - W^2} = \frac{60\,\mathsf{N}}{2\mathsf{N}/\mathsf{m}} = 30\,\mathsf{m}.$$

Thus (16c.2) assures
$$\mathbf{z} = \mathsf{Q}\cdot\operatorname{asinh}\left(\frac{L}{\mathsf{Q}}\right) = 30\,\mathsf{m}\cdot\operatorname{asinh}\left(\frac{40}{30}\right) \lessapprox 33\,\mathsf{m}.$$

So the span is $2\mathbf{z} \lessapprox 66\,\mathsf{m}$. ♦

17.1: Unequal poles. *We have pole-0 and pole-1 of heights $0 \le V_0, V_1$, not both zero. Running between is a length-Λ cable, long enough that the vertex lies between the poles; just touching the ground. For k=0,1, use $\boldsymbol{\ell}_k$ for the arclength from pole-k to vertex, and $\mathbf{z}_k$ for the horizontal distance. Compute $\boldsymbol{\ell}_0$.* ◊

Prelim. Define height difference $D := V_1 - V_0$. □

17.2: Theorem. *When $V_1 \ne V_0$, the $\boldsymbol{\ell}_0$ arclength is*

$$†: \quad \boldsymbol{\ell}_0 = \tfrac{1}{D}\Big[\sqrt{V_1 V_0 \cdot [\Lambda^2 - D^2]} - V_0\Lambda\Big]. \qquad ◊$$

Plausibility. Exchanging subscripts gives

$$‡: \quad \boldsymbol{\ell}_1 = \tfrac{1}{-D}\Big[\sqrt{V_0 V_1 \cdot [\Lambda^2 - [-D]^2]} - V_1\Lambda\Big].$$

Adding (‡) to (†) shows that

$$£: \quad \boldsymbol{\ell}_1 + \boldsymbol{\ell}_0 \overset{\text{note}}{=\!=} \tfrac{1}{D}[V_1\Lambda - V_0\Lambda] \overset{\checkmark}{=} \Lambda.$$

We now vary a V_k, which will vary D. Must it also vary Λ [making derivatives harder to calculate]? *No!* As (†) does *not* directly mention either $\mathbf{z}_k$, we can vary pole-separation to keep Λ constant [with vertex touching the ground].

Setting $V_0 = 0$ [i.e, the vertex is at pole-0] gives

$$\boldsymbol{\ell}_0\big|_{V_0=0} = \tfrac{1}{V_1}\Big[\overbrace{\sqrt{0} - 0}^{\text{units of } @^2}\Big] \overset{\checkmark}{=} 0.$$

[No gain to setting V_1=0 in (†), as (£) shows we will get Λ.] □

Sending $V_1 \to V_0$. [Our derivation of (†) uses $V_1 \ne V_0$, so we need to take a limit.] The limit has the poles of equal height, so we *expect* that *the limit-value of $\boldsymbol{\ell}_0$ is $\Lambda/2$.*

Since $D = V_1 - V_0$, derivative $\frac{dD}{dV_1} = 1 = \frac{dV_1}{dV_1}$. Let P denote $V_1 V_0 \cdot [\Lambda^2 - D^2]$. Then l'Hôpital's tells us that $\lim_{V_1 \to V_0} \boldsymbol{\ell}_0$ equals the limit of ratio

$$*: \quad \frac{\frac{d}{dV_1}[\sqrt{P} - \Lambda V_0]}{\frac{d}{dV_1} D} \overset{\text{note}}{=\!=} \frac{\frac{d}{dV_1}[\sqrt{P}]}{1} \overset{\text{Chain}}{\underset{\text{rule}}{=\!=}} \frac{1}{2\sqrt{P}} \cdot \frac{dP}{dV_1}.$$

Note $\frac{dP}{dV_1} = V_0[\Lambda^2 - D^2] + V_1 V_0 \cdot [0 - 2D]$. Thus

$$\lim_{V_1 \to V_0} \frac{dP}{dV_1} = V_0[\Lambda^2 - 0^2] - 0 = V_0 \cdot \Lambda^2.$$

Also, $\lim_{V_1 \to V_0} P = V_0 V_0 \cdot \Lambda^2$, so $\lim_{V_1 \to V_0} \sqrt{P} = V_0 \Lambda$. Thus the limit of RhS(∗) equals

$$\frac{1}{2 \cdot V_0 \Lambda} \cdot V_0 \cdot \Lambda^2 \overset{\checkmark}{=} \frac{\Lambda}{2},$$

as predicted. □

Unequal soln. From (16a), the cable's shape[♡11] is

$$h(x) = \frac{1}{\mathbf{r}} \cdot [\cosh(\mathbf{r}x) - 1].$$

This, and (16b), yield

$$\begin{aligned} \mathbf{r}V_k + 1 &= \cosh(\mathbf{r}\cdot\mathbf{z}_k) \quad \textit{and} \\ \mathbf{r}\,\boldsymbol{\ell}_k &= \sinh(\mathbf{r}\cdot\mathbf{z}_k)\,. \end{aligned}$$

Courtesy Pythagorus

$$1^2 = [\mathbf{r}V_k + 1]^2 - [\mathbf{r}\boldsymbol{\ell}_k]^2\,.$$

Subtracting 1 from both sides, then dividing by $\mathbf{r}$, yields $0 = \mathbf{r}{V_k}^2 + 2V_k - \mathbf{r}\cdot{\boldsymbol{\ell}_k}^2$. Solving for $\mathbf{r}$,

$$\frac{1}{\mathbf{r}} = \frac{{\boldsymbol{\ell}_k}^2 - {V_k}^2}{2V_k}\,.$$

Thus $\frac{{\boldsymbol{\ell}_0}^2 - {V_0}^2}{V_0} = \frac{{\boldsymbol{\ell}_1}^2 - {V_1}^2}{V_1}$. Cross-multiplying, then subtracting,

$$V_1{\boldsymbol{\ell}_0}^2 - V_0{\boldsymbol{\ell}_1}^2 + \overbrace{V_0{V_1}^2 - V_1{V_0}^2}^{= V_0V_1D} = 0\,.$$

Since $\boldsymbol{\ell}_1 = \Lambda - \boldsymbol{\ell}_0$, our $\boldsymbol{\ell}_0$ is a root of polynomial

$$\begin{aligned} f(t) &:= V_1t^2 - V_0[\Lambda - t]^2 + V_0V_1D \\ &= Dt^2 + 2V_0\Lambda\cdot t + V_0[V_1D - \Lambda^2]\,. \end{aligned}$$

Computing the polynomial's discriminant,

$$\begin{aligned} \tfrac{1}{4}\mathrm{Discr}(f) &= \tfrac{1}{4} \cdot \Big[[2V_0\Lambda]^2 - 4\cdot D\cdot V_0[V_1D - \Lambda^2]\Big] \\ &= V_0\Big[V_0\Lambda^2 - D[V_1D - \Lambda^2]\Big] \\ &= V_0\Big[[V_0 + D]\Lambda^2 - V_1D^2\Big] \\ &= V_0[V_1\Lambda^2 - V_1D^2] \\ &= V_1V_0[\Lambda^2 - D^2]\,. \end{aligned}$$

The roots of f are

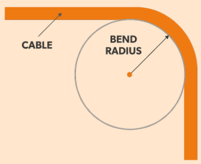
$$\begin{aligned} &\tfrac{1}{2D} \cdot \Big[-2V_0\Lambda \pm 2\sqrt{V_1V_0[\Lambda^2 - D^2]}\Big] \\ &= \tfrac{1}{D} \cdot \Big[\pm\sqrt{V_1V_0[\Lambda^2 - D^2]} - V_0\Lambda\Big]. \end{aligned}$$

Our $\boldsymbol{\ell}_0$ is non-negative, hence (†).

[♡11] I've made $\mathbf{r} \stackrel{\text{def}}{=} \frac{\text{vertex-density}}{\text{vertex-tension}}$ lower-case, as it is currently unknown.

> *Difficulties mastered are opportunities won.*
> *–Winston Churchill*

18.1: Bent outta shape? *A 22m cable, whose minimum bending radius (see W: Bending radius), is 3m*



has its two ends attached to a track in the ceiling of a workshop. Bringing the ends together lowers the cable-vertex. How low can the vertex be before transgressing min–Bend-radius somewhere along the cable? ♢

Curvature. Consider an oriented-curve $\mathcal{C}$, point $\mathbf{P}_0$ on $\mathcal{C}$, and have $P(s)$ be the point along $\mathcal{C}$ at arclength distance s from $\mathbf{P}_0$. [So $s ::$ ⓓ.] Relative to horizontal, let $\theta(s) = \theta_{\mathcal{C}}(s)$ be the angle of the tangent-line at $P(s)$. Writing

$$†: \quad \theta(s) = \text{Some-Particular-Formula}(s)$$

is sometimes called a ***Whewell equation*** for the curve (William Whewell, pron. "Hu-well"). Its derivative w.r.t arclength,

$$‡: \quad \kappa(s) = \kappa_{\mathcal{C}}(s) := \theta'(s),$$

gives the *curvature* at $P(s)$. This (‡) is called a ***Cesàro equation*** (Ernesto Cesàro) for $\mathcal{C}$. As we expect, $\kappa(s) :: \frac{1}{ⓓ}$, since curvature is the reciprocal of radius-of-curvature. □

Prelim. [This "Bent" problem is kinda hokey, as our derviation of a hanging-cable assumed ∞ flexibility, whence min-Bend-radius should be zero. But we proceed anyway...]

We seek the curvature of cable $h(x) = \mathbf{Q}\cosh(\frac{x}{\mathbf{Q}})$. We *could* use the annoying Calc-I formula

$$\kappa = \frac{h''}{[1 + h'^2]^{3/2}}.$$

More natural and elegant is derive a Cesàro formula for our beloved catenary. □

18.2: Catenary curvature lemma. *Consider catenary*

$$h(x) = \mathbf{Q}\cosh(\tfrac{x}{\mathbf{Q}}),$$

where $x, \mathbf{Q} ::$ ⓓ. Measuring from the vertex by arclength s,

$$\text{18.3:} \quad \theta_{cat}(s) = \arctan(s/\mathbf{Q}) \quad \textit{and}$$

$$\text{18.4:} \quad \kappa_{cat}(s) = \frac{\mathbf{Q}}{s^2 + \mathbf{Q}^2}$$

are the Whewell and Cesàro formulae, respectively. ♢

Proof. Previous work shows that

$$\textit{Slope} = h'(x) = \sinh(\tfrac{x}{\mathbf{Q}}) \quad \textit{and}$$
$$\textit{Arclength} = s(x) = \mathbf{Q}\sinh(\tfrac{x}{\mathbf{Q}}).$$

Thus $s/\mathbf{Q}$ gives slope ITOf arclength. Hence (18.3) is the *angle* at arclength s. Differentiation and algebra produces (18.4). ♦

Bent soln. With $L := \frac{22\text{m}}{2}$ half the cable-len, and min Bend-radius $\mathsf{B} := 3\text{m}$, I claim max-vertical-drop is

$$\text{18.5:} \quad V_{\text{Max}} = \sqrt{L^2 + \mathsf{B}^2} - \mathsf{B}$$
$$= [\sqrt{11^2 + 3^2} - 3]\,\text{m} \approx 8.4\,\text{m}.$$

[*Were min-Bend-radius* <u>*zero*</u>*, we'd expect the max drop to be the cable going straight down, then straight back up again. And indeed,* $V_{\text{Max}}(0\text{m}) = L$.] *Here's the argument for* (18.5):

Formula (18.4) says max-curvature occurs at the vertex (unsurprisingly), so the min radius-of-curve is $\mathbf{Q}$.

[*The next time I teach this course, I will exchange names* $\mathbf{R}$ *and* $\mathbf{Q}$*, making* $\mathbf{R}$ *min-radius-of-curve.*]

We seek to maximize ceiling-to-vertex vertical drop, v, without violating min-Bend-radius. As formula (16c.1) gives

$$\mathbf{Q}(v) = \frac{L^2 - v^2}{2v},$$

we maximize v such that $\mathbf{Q}(v) \geq \mathsf{B}$. [*The graph of* $\mathbf{Q}(v) = \frac{L^2/2}{v} - \frac{1}{2}v$ *is a hyperbola with one asymptote vertical* [send $v \to 0$] *and the other with slope* $\frac{-1}{2}$ [send $v \to \infty$]*. This hyperbola twice intersects the horiz-line at height-*$\mathbf{Q}$*: At a negative value less than* $-L$*, and* (the value we seek) *at a positive value less than* L.] Rewrite inequality $\mathsf{B} \leq \mathbf{Q}(v)$ as

$$v^2 + 2\mathsf{B}v - L^2 \leq 0.$$

As a fnc-of-v the poly's discriminant is $2^2[\mathsf{B}^2 + L^2]$, whence its roots $\pm\sqrt{L^2 + \mathsf{B}^2} - \mathsf{B}$. *Thus* (18.5). ♦

> *To a man who has only a hammer, every problem looks like a nail.* –*Mark Twain* (paraphrased)

Convolutions [Chap4–NSS9, P.237.]

Recall the ***identity fnc*** $Id := [t \mapsto t]$. So $Id^3(x) = x^3$, and Id^0 is the constant-fnc $\mathbf{1}$. Below, let $\mathbb{J} := [0, \infty)$.

Convolution defn. Given (locally-integrable) fncs $\mathbf{f}, \mathbf{g}: \mathbb{J} \to \mathbb{C}$, their one-sided ***convolution*** is the fnc mapping $\mathbb{J} \to \mathbb{C}$ by

19.1: $$[\mathbf{f} \circledast \mathbf{g}](t) := \int_0^t \mathbf{f}(t-v) \cdot \mathbf{g}(v)\, \mathrm{d}v\,.$$

Easily, we get these algebraic properties:

19.2: *Convolution is commutative and associative. Convolution is bilinear*[♡1], *in that*
$$[\mathbf{f}_1 + \mathbf{f}_2] \circledast \mathbf{g} = [\mathbf{f}_1 \circledast \mathbf{g}] + [\mathbf{f}_2 \circledast \mathbf{g}]\,,$$
$$[5\mathbf{f}] \circledast \mathbf{g} = 5 \cdot [\mathbf{f} \circledast \mathbf{g}]\,,$$
for arb. fncs $\mathbf{f}, \mathbf{f}_1, \mathbf{f}_2$ *and arbitrary scalar,* 5.
Convolution commutes with complex-conjugation: $\overline{f \circledast g} = \overline{f} \circledast \overline{g}$.

We also have this cty property [more is true]:

19.3: If $\mathbf{f}$,$\mathbf{g}$ continuous, then $[\mathbf{f} \circledast \mathbf{g}]$ is cts.

CAVEAT: We do *not* have a formula for how convolution interacts with multiplication; we have no nice formula for $F \circledast [g \cdot h]$.

Powers. As a shorthand, the "n^{th} ***convolution power*** of $\mathbf{f}$",
$$\mathbf{f}^{\circledast n} := \mathbf{f} \circledast \mathbf{f} \circledast \overset{n}{\cdots} \circledast \mathbf{f}\,,$$
is the result of convolving together n copies of $\mathbf{f}$. In particular, $\mathbf{1}^{\circledast[n+1]}$ is the n^{th}-antideriv of $\mathbf{1}$ (i.e, x^0) whose derivatives are zero at the origin. So

20a: $$\mathbf{1}^{\circledast[n+1]} = \tfrac{1}{n!} \cdot Id^n \overset{\text{i.e}}{=\!=} \Big[x \mapsto \frac{x^n}{n!}\Big]\,.$$

We get this nice corollary.

[♡1] In the other order, $f \circledast [\mathbf{g}_1 + \mathbf{g}_2] = [f \circledast \mathbf{g}_1] + [f \circledast \mathbf{g}_2]$; in other words: *"Convolution distributes over addition"*. Also, $f \circledast [7g] = 7[f \circledast g]$; i.e: *"Scalars factor-out"*.

20b: Power-of-x Lemma. *Consider a continuous function* $\boldsymbol{\beta}: \mathbb{J} \to \mathbb{C}$, *and a natnum* N. *Then*

$\dagger_N$: $$\big[\tfrac{1}{N!} \cdot Id^N\big] \circledast \boldsymbol{\beta} = B_N\,,$$

where B_N *is the unique function such that*

$\ddagger$: $$0 = B_N(0) = B_N'(0) = B_N''(0) = \ldots = B_N^{(N)}(0)\,.$$

and $B_N^{(N+1)} = \boldsymbol{\beta}$. ◊

Proof. For an arbitrary fnc $\mathbf{g}$, the FTC says that $[\mathbf{1} \circledast \mathbf{g}](t) \overset{\text{def}}{=\!=} \int_0^t \mathbf{g}$ is the antideriv G of $\mathbf{g}$ such that $G(0) = 0$. Courtesy (20a), our $\big[\frac{1}{N!} \cdot Id^N\big] \circledast \boldsymbol{\beta}$ is
$$\mathbf{1} \circledast [\mathbf{1} \circledast \overset{N+1}{\cdots} [\mathbf{1} \circledast \boldsymbol{\beta}] \ldots]\,,$$
using the associativity of convolution. Hence $\big[\frac{1}{N!} \cdot Id^N\big] \circledast \boldsymbol{\beta}$ is indeed the B_N defined by ($\ddagger$). ♦

Alt Pf. Just for fun, here is an alternate proof using a derivative-of-convolution formula, (24e), that we'll shortly deduce.

Defining $\boldsymbol{\alpha}_k(t) = t^k/k!$, note $[\boldsymbol{\alpha}_{k+1}]' = \boldsymbol{\alpha}_k$. Fix a natnum K satisfying ($\dagger_K$). Differentiating,
$$[\boldsymbol{\alpha}_{K+1} \circledast \boldsymbol{\beta}]' \overset{\text{by (24e)}}{=\!=} \Big[[\boldsymbol{\alpha}_{K+1}]' \circledast \boldsymbol{\beta}\Big] + \Big[\boldsymbol{\alpha}_{K+1}(0) \cdot \boldsymbol{\beta}\Big]$$
$$= [\boldsymbol{\alpha}_K \circledast \boldsymbol{\beta}]\,,$$
since $\boldsymbol{\alpha}_{K+1}(0)$ is 0, as $K+1$ is positive. So $[\boldsymbol{\alpha}_{K+1} \circledast \boldsymbol{\beta}]'$ is B_K. Thus
$$[\boldsymbol{\alpha}_{K+1} \circledast \boldsymbol{\beta}](t) = \int_0^t B_K \overset{\text{by FTC}}{=\!=} B_{K+1}(t)\,.$$
Hence ($\dagger_{K+1}$). We've shown that ($\dagger_K$) $\Rightarrow$ ($\dagger_{K+1}$), as desired. ♦

Ex.C1. Note that $\frac{\mathrm{d}}{\mathrm{d}v}([5+1-v] \cdot \mathrm{e}^v) = [5-v] \cdot \mathrm{e}^v$. So
$$[Id \circledast \exp](t) \overset{\text{def}}{=\!=} \int_0^t [t-v] \cdot \mathrm{e}^v\, \mathrm{d}v = [[t+1-v] \cdot \mathrm{e}^v]\Big|_{v=0}^{v=t}$$
$$= \mathrm{e}^t - [t+1]\,. \quad \square$$

Ex.C2. Let $f(x) := x^2$ and $\boldsymbol{\beta}(x) := 30[x^4 + x]$. Then

$$[f \circledast \boldsymbol{\beta}](t) = \int_0^t [t-v]^2 \cdot 30[v^4 + v]\,\mathrm{d}v\,.$$

The integrand is a poly, which we could multiply-out, then integrate. Alternatively, cheerfully apply ($\dagger_2$), and antidiff $\boldsymbol{\beta}$ thrice to get

$$\frac{30\,x^7}{5{\cdot}6{\cdot}7} + \frac{30\,x^4}{2{\cdot}3{\cdot}4} = \frac{x^7}{7} + \frac{5x^4}{4}\,.$$

Multiply by 2! to conclude that

$$[f \circledast \boldsymbol{\beta}](t) = \frac{2}{7}{\cdot}t^7 + \frac{5}{2}{\cdot}t^4\,. \qquad \square$$

Ex.C3. Let's convolve exponentials $f(x) := \mathrm{e}^{Bx}$ and $g(x) := \mathrm{e}^{Cx}$, where $B,C \in \mathbb{C}$.

CASE: $B = C$ The integrand for computing $[f \circledast f](5)$ is $\mathrm{e}^{B[5-v]} \cdot \mathrm{e}^{Bv} \stackrel{\text{note}}{=\!=} \mathrm{e}^{B\cdot 5}$; constant. Its integral is thus $5 \cdot \mathrm{e}^{B\cdot 5}$. Hence

21a: $$[x \mapsto \mathrm{e}^{Bx}]^{\circledast 2}(t) = [f \circledast f](t) = t \cdot \mathrm{e}^{Bt}\,.$$
In functional notation, $f \circledast f = \mathit{Id} \cdot f\,.$

[In the B=0 case, this says $\mathbf{1} \circledast \mathbf{1} = \mathit{Id}$, which is indeed correct.]

CASE: $B \neq C$ Define difference $D := C - B$. The $[f \circledast g](5)$ integrand is $\mathrm{e}^{B[5-v]} \cdot \mathrm{e}^{Cv} \stackrel{\text{note}}{=\!=} \mathrm{e}^{B\cdot 5} \cdot \mathrm{e}^{D\cdot v}$. Its integral is $\frac{\mathrm{e}^{B\cdot 5}}{D} \cdot \mathrm{e}^{D\cdot v}\big|_{v=0}^{v=5}$, i.e, $\frac{\mathrm{e}^{B\cdot 5}}{D} \cdot [\mathrm{e}^{D\cdot 5} - 1]$.

This equals $\frac{1}{D}[\mathrm{e}^{C\cdot 5} - \mathrm{e}^{B\cdot 5}]$. Consequently,

21b: $$[f \circledast g](t) = \frac{[\mathrm{e}^{Ct} - \mathrm{e}^{Bt}]}{C - B} \stackrel{\text{note}}{=\!=} \frac{[\mathrm{e}^{Bt} - \mathrm{e}^{Ct}]}{B - C}\,.$$
$$\textit{I.e,}\quad f \circledast g = \frac{g - f}{C - B} = \frac{f - g}{B - C}\,.$$

This is symmetric in B and C, as it must be. $\square$

A shorthand. I'll write '$[9x] \circledast \mathrm{e}^{3x}$ equals...' to mean:

Let $f(u) := 9u$ *and* $g(z) := \mathrm{e}^z$.
Then $[f \circledast g](x)$ *equals...*

I.e, I will sometimes use the same letter for the input-vars, and the output-var. $\square$

Ex.C4.1. We seek to compute $H := [9x] \circledast \mathrm{e}^{3x}$.

Let's solve this just by using properties of convolution. Let $\mathbf{G} := \mathrm{e}^{3x}$. Since $[\int 3\mathbf{G}] = \mathbf{G} + \text{Const}$, and $\mathbf{G}\rfloor_{x=0}$ is 1, it follows that

$$\dagger: \qquad 1 \circledast [3\mathbf{G}] = \mathbf{G} - 1\,.$$

Since convolution is bilinear,

$$\begin{aligned} H = 9[x \circledast \mathbf{G}] &= x \circledast [9\mathbf{G}] \\ &= [1 \circledast 1] \circledast [9\mathbf{G}] \\ &= 1 \circledast [1 \circledast [9\mathbf{G}]]\,, \end{aligned}$$

since $\circledast$ is associative. Computing the inside-convolution,

$$1 \circledast [9\mathbf{G}] = 3{\cdot}[1 \circledast [3\mathbf{G}]] \stackrel{\text{by }(\dagger)}{=\!=} 3{\cdot}[\mathbf{G} - 1] = 3\mathbf{G} - 3\,.$$

$$\begin{aligned} \textit{So,}\quad H &= 1 \circledast [3\mathbf{G} - 3{\cdot}1] \\ &= [1 \circledast 3\mathbf{G}] - 3{\cdot}[1 \circledast 1] \\ &= [\mathbf{G} - 1] - 3x = \mathrm{e}^{3x} - 1 - 3x\,. \end{aligned} \qquad \square$$

Ex.C4.2. The preceding example showed that

$$\ddagger: \qquad \begin{aligned} \mathbf{1} \circledast \mathbf{G} &= \tfrac{1}{3}[\mathbf{G} - 1]\,, \quad \text{and} \\ \mathbf{1}^{\circledast 2} \circledast \mathbf{G} &= \tfrac{1}{9}[\mathbf{G} - 1 - 3x]\,. \end{aligned}$$

Continuing, $[\mathbf{1}^{\circledast 3} \circledast \mathbf{G}]$ is *one-ninth* of

$$\begin{aligned} &[\mathbf{1} \circledast \mathbf{G}] - [\mathbf{1} \circledast 1] - 3[\mathbf{1} \circledast x] \\ &= \tfrac{1}{3}[\mathbf{G} - 1] - x - 3{\cdot}\frac{x^2}{2} \\ &= \tfrac{1}{3}\Big[\mathbf{G} - 1 - 3x - 3^2{\cdot}\frac{x^2}{2}\Big] \\ &\stackrel{\text{note}}{=\!=} \tfrac{1}{3}\Big[\mathbf{G} - \frac{[3x]^0}{0!} - \frac{[3x]^1}{1!} - \frac{[3x]^2}{2!}\Big]\,. \end{aligned}$$

Hence

$$\mathbf{1}^{\circledast 3} \circledast \mathbf{G} = \frac{1}{27} \cdot \Big[\mathbf{G} - \Big[\frac{[3x]^0}{0!} + \frac{[3x]^1}{1!} + \frac{[3x]^2}{2!}\Big]\Big].$$

The pattern is clear:

For each natnum N, with $\mathbf{G}$ *denoting* e^{3x},

22a: $$\begin{aligned} \tfrac{1}{N!}{\cdot}[x^N \circledast \mathbf{G}] &\stackrel{\text{recall}}{=\!=} \mathbf{1}^{\circledast[N+1]} \circledast \mathbf{G} \\ &= \frac{1}{3^{N+1}}\Big[\mathbf{G} - \sum_{k=0}^{N} \frac{[3x]^k}{k!}\Big]. \end{aligned}$$

Rewriting,

$$\text{22b:}\quad x^N \circledast e^{3x} \;=\; \frac{N!}{3^{N+1}}\left[e^{3x} - \sum_{k=0}^{N}\frac{[3x]^k}{k!}\right].$$

The above sum, $\sum_{k=0}^{N}\frac{[3x]^k}{k!}$, we recognize as the *$N^{th}$-Maclaurin-polynomial of* e^{3x}; see below. Before generalizing this result, let us compute an example with [*shudder*] *actual numbers*.

Let $R := [6 - 9x + 54x^2] \circledast e^{3x}$. Then

$$R \;=\; 6\cdot[1 \circledast \mathbf{G}] - 9\cdot[x \circledast \mathbf{G}] + 54\cdot[x^2 \circledast \mathbf{G}].$$

From (22b), or (‡), note

$$6\cdot[1 \circledast \mathbf{G}] = 6\cdot\tfrac{1}{3}\cdot[\mathbf{G}-1] = 2\mathbf{G}-2, \quad \textit{and}$$
$$-9\cdot[x \circledast \mathbf{G}] = -9\cdot\tfrac{1}{9}\cdot[\mathbf{G}-1-3x] = -\mathbf{G}+1+3x, \; \textit{and}$$
$$54\cdot[x^2 \circledast \mathbf{G}] = 54\cdot\tfrac{2!}{3^3}\cdot[\text{Terms}] = 4\Big[\mathbf{G}-1-3x-\tfrac{9}{2}x^2\Big].$$

Adding these together says that

$$R \;=\; 5e^{3x} - [5 + 9x + 18x^2]. \qquad \Box$$

Maclaurin polynomial. For a natnum N, consider a function G which is N-times differentiable. Then the "N^{th} ***Maclaurin polynomial*** of G" is the unique polynomial p of $\text{Deg}(p) \le N$, whose first $N{+}1$ derivatives agree with G's at the origin. I.e

$$p(0) = G(0),\; p'(0) = G'(0),\; p''(0) = G''(0),$$
$$\ldots, p^{(N-1)}(0) = G^{(N-1)}(0),\; p^{(N)}(0) = G^{(N)}(0).$$

An explicit formula for p is

$$p(x) \;:=\; \sum_{k=0}^{N}\frac{G^{(k)}(0)}{k!}\cdot x^k.$$

Use $\text{Mac}_{G,N}$ to denote this p; it is the N^{th} ***Maclaurin polynomial of*** G. $\Box$

23: Convolve-Mac Thm. *Consider an integrable fnc $\boldsymbol\beta$ on $[0,\infty)$, and fix a natnum N. Let $g = g_N$ be a fnc whose $[N{+}1]^{st}$-derivative is $\boldsymbol\beta$, i.e, $g^{(N+1)} = \boldsymbol\beta$. Then*

$$\mathbf{1}^{\circledast[N+1]} \circledast \boldsymbol\beta \;=\; g - \text{Mac}_{g,N}. \qquad \Diamond$$

Proof. This follows immediately from Power-of-x Lemma, (20b), on page 40. ♦

Convolve-Mac 1. Compute $f := [x^5 \circledast \cos(2x)]$.

C-M-Soln. With $\boldsymbol\beta := \cos(2x)$, note $\frac{f}{5!} = \frac{x^5}{5!} \circledast \boldsymbol\beta$, so

$$f \;=\; 5!\cdot[\mathbf{1}^{\circledast 6} \circledast \boldsymbol\beta].$$

A particular 6^{th}-antideriv of $\boldsymbol\beta$ is

$$g \;:=\; -\cos(2x)/2^6 \overset{\text{note}}{=\!=} -\boldsymbol\beta/2^6.$$

Recall $\cos(t) = 1 - \frac{t^2}{2} + \frac{t^4}{24} - \frac{t^6}{6!} + \ldots$. Plugging in $2x$ for t shows $1 - 2x^2 + \frac{2}{3}x^4 - \ldots$ is the Mac-series for $\boldsymbol\beta$. Hence $\text{Mac}_{\boldsymbol\beta,5} = [1 - 2x^2 + \frac{2}{3}x^4]$. Finally,

$$f \;=\; 5!\cdot[g - \text{Mac}_{g,5}] \;=\; -\tfrac{5!}{2^6}\cdot[\boldsymbol\beta - \text{Mac}_{\boldsymbol\beta,5}]$$
$$=\; \tfrac{5\cdot 3}{2^3}\cdot[\text{Mac}_{\boldsymbol\beta,5} - \boldsymbol\beta]$$
$$=\; \tfrac{5}{8}\cdot\Big[[3 - 6x^2 + 2x^4] - 3\cos(2x)\Big]. \qquad ♦$$

Derivative notation. Below, for a two-variable function $H(x,y)$, we use $H_1()$ to mean the partial-derivative w.r.t the 1^{st} variable; so $H_1()$ is a synonym for $H_x()$. And $H_2()$ is $H_y()$. $\Box$

24a: Chain-rule Lemma. *Consider equations*

$$x = \boldsymbol\alpha(t) \quad \textit{and} \quad y = \boldsymbol\beta(t) \quad \textit{and} \quad z = H(x,y),$$

for differentiable functions $\boldsymbol\alpha, \boldsymbol\beta, H$. Then composition $\varphi(t) := H(\boldsymbol\alpha(t), \boldsymbol\beta(t))$ is differentiable. Moreover,

$$\text{24b:}\quad \frac{dz}{dt} \;=\; \frac{dz}{dx}\cdot\frac{dx}{dt} + \frac{dz}{dy}\cdot\frac{dy}{dt}; \quad [\textit{Leibniz}]$$
$$\varphi'(t) \;=\; H_1(\boldsymbol\alpha(t),\boldsymbol\beta(t))\cdot\boldsymbol\alpha'(t) + H_2(\boldsymbol\alpha(t),\boldsymbol\beta(t))\cdot\boldsymbol\beta'(t), \quad [\textit{Newton}]$$

where Leibniz names the variables, and Newton names the functions. $\Diamond$

24c: DUI: Differentiation under Integral. *Consider fnc $G(x,v)$ defined on a rectangle $\mathbf{U} := [x_0,x_1] \times [v_0,v_1]$ in the plane. Suppose partial-deriv $G_1()$ is cts on $\mathbf{U}$. Then for arb. values, say, 3 and 5, in $[v_0 \,..\, v_1]$, the fnc*

$$H(x) \;:=\; \int_3^5 G(x,v)\,dv$$

is differentiable, and

$$H'(x) \;=\; \int_3^5 G_1(x,v)\,dv. \qquad \Diamond$$

Proof. Fix, say, $x{=}7$. From a non-zero ε, form difference quotient

$$\frac{H(7+\varepsilon)-H(7)}{\varepsilon} \;=\; \int_3^5 \frac{G(7+\varepsilon,v)-G(7,v)}{\varepsilon}\,\mathrm{d}v\,.$$

Send $\varepsilon \to 0$. In order to pass that limit through the integral sign, note the following. Since $G_1(\cdot,\cdot)$ is cts on compact set $\mathbf{U}$, our $G(\cdot,\cdot)$ is uniformly Lipschitz in the x-direction. Hence we can use the Dominated Convergence thm to commute the limits. ♦

24d: Leibniz-rule Lemma. *Consider continuous function $G{:}\mathbb{J}{\times}\mathbb{J}{\to}\mathbb{R}$; hence $G_1()$ is cts. Define*

24d∗: $$H(x,y) \;:=\; \int_0^y G(x,v)\,\mathrm{d}v\,.$$

Then $\varphi(t) := H(t,t)$ is diff'able, and

24d†: $$\varphi'(t) \;=\; \Big[\int_0^t G_1(t,v)\,\mathrm{d}v\Big] + G(t,t)\,. \qquad \diamondsuit$$

Proof. Notice that $\varphi(t) = H(\boldsymbol{\alpha}(t),\boldsymbol{\beta}(t))$, where fncs $\boldsymbol{\alpha}(t) := t =: \boldsymbol{\beta}(t)$. Applying the Chain rule (24b),

$$\begin{aligned}\varphi'(t) \;&=\; H_1(t,t)\cdot\tfrac{\mathrm{d}t}{\mathrm{d}t} + H_2(t,t)\cdot\tfrac{\mathrm{d}t}{\mathrm{d}t}\\ &=\; H_1(t,t) + H_2(t,t)\,.\end{aligned}$$

By DUI (24c), our $H_1(x,y) = \int_0^y G_1(x,v)\,\mathrm{d}v$. Hence

$$H_1(t,t) \;=\; \int_0^t G_1(t,v)\,\mathrm{d}v\,.$$

By FTC, moreover, $H_2(x,t) = G(x,t)$. Thus

$$H_2(t,t) = G(t,t)\,.$$

These three displays, together, yield (24d†). ♦

24e: Leibniz corollary. *Suppose $\boldsymbol{\alpha},\boldsymbol{\beta}$ are differentiable fncs on $\mathbb{J}$. Then $[\boldsymbol{\alpha}\circledast\boldsymbol{\beta}]$ is differentiable,* [♡12] *and*

24d‡: $$\begin{aligned}[\boldsymbol{\alpha}\circledast\boldsymbol{\beta}]'(t) \;&=\; [\boldsymbol{\alpha}'\circledast\boldsymbol{\beta}](t) + \boldsymbol{\alpha}(0)\cdot\boldsymbol{\beta}(t)\\ &\underset{\text{by symmetry}}{=\!=\!=}\; [\boldsymbol{\alpha}\circledast\boldsymbol{\beta}'](t) + \boldsymbol{\alpha}(t)\cdot\boldsymbol{\beta}(0)\,.\end{aligned} \qquad \diamondsuit$$

Proof. Define $G(x,v) := \boldsymbol{\alpha}(x-v)\cdot\boldsymbol{\beta}(v)$, then H as in (24d∗). So $[\boldsymbol{\alpha}\circledast\boldsymbol{\beta}](t) \overset{\text{def}}{=\!=} H(t,t)$. Using that $G(t,t) = \boldsymbol{\alpha}(0)\cdot\boldsymbol{\beta}(t)$, applying (24d†) yields (24d‡). ♦

[♡12] Wikipedia gives a slightly different formula, but for the derivative of a 2-sided convolution. Our 1-sided convolution has an edge-effect when differentiated.

24f: Convol-diff Thm. *Fix a natnum N. Consider an $\mathbf{f} \in \mathbf{C}^N$ and $\mathbf{g} \in \mathbf{C}^{N-1}$. [When $N = 0$, we just need $\mathbf{g}$ locally-integrable.] Then $\mathbf{f}\circledast\mathbf{g}$ is in $\mathbf{C}^N$, and*

P_N: $$[\mathbf{f}\circledast\mathbf{g}]^{(N)} \;=\; [\mathbf{f}^{(N)}\circledast\mathbf{g}] + \sum_{j+k\,=\,N-1} \mathbf{f}^{(j)}(0)\cdot\mathbf{g}^{(k)}\,,$$

where the sum [♡13] *is taken over all <u>ordered</u> pairs (j,k) of natnums.* ♢

Proof. For $N{=}0$, this says $[\mathbf{f}\circledast\mathbf{g}] = [\mathbf{f}\circledast\mathbf{g}]$; true.

Now fix an N for which (P_N) holds. We differentiate RhS(P_N), by setting $\boldsymbol{\alpha} := \mathbf{f}^{(N)}$ and $\boldsymbol{\beta} := \mathbf{g}$, and applying (24d‡). It yields that $[\mathbf{f}\circledast\mathbf{g}]^{(N+1)}$ equals

$$[\boldsymbol{\alpha}'\circledast\boldsymbol{\beta}](t) + \boldsymbol{\alpha}(0)\cdot\boldsymbol{\beta}(t) + \sum_{j+\ell\,=\,N-1} \mathbf{f}^{(j)}(0)\cdot\mathbf{g}^{(\ell+1)}(t)\,,$$

summed over ordered-pairs (j,ℓ) of natnums. Setting $k := \ell{+}1$, we can re-write this as

$$[\boldsymbol{\alpha}'\circledast\boldsymbol{\beta}](t) + \sum_{j+k\,=\,N} \mathbf{f}^{(j)}(0)\cdot\mathbf{g}^{(k)}(t)\,.$$

Noting that $\boldsymbol{\alpha}'$ is $\mathbf{f}^{(N+1)}$, gives (P_{N+1}). ♦

[♡13] E.g $[\mathbf{f}\circledast\mathbf{g}]''(7)$ equals $[\mathbf{f}''\circledast\mathbf{g}](7)$ plus $\mathbf{f}'(0)\mathbf{g}(7) + \mathbf{f}(0)\mathbf{g}'(7)$.

Convol-GenTar Algorithm

[See P.237 of **NSS9**.] A polynomial

$$q(z) := C_N z^N + \ldots + C_1 z^1 + C_0 z^0 ,$$

with $C_N \neq 0$, hands us an operator $\mathsf{L} := q(\mathbf{D})$. We seek a fnc $y{=}y(t)$ solving DE

25a: $$\mathsf{L}(y) = G ,$$

for a given target fnc G.

1st-step. Use CCLDE to produce a function f solving ZeroTar $\mathsf{L}(f) = 0$, with initial conditions

25b: $$f^{(N-1)}(0) = 1/C_N , \quad \textit{and}$$
$$0 = f(0) = f'(0) = \ldots f^{(N-2)}(0) .$$

2nd-step. Compute $y := f \circledast G$.

What's the magic behind Convol-GenTar algorithm? To see that y solves (25a) note, because of initial conditions (25b), that we have this:

$$\text{For } j = 0, 1, \ldots, N{-}1: \quad y^{(j)} = [\mathbf{f}^{(j)} \circledast G] .$$

$$\text{And} \quad y^{(N)} = [\mathbf{f}^{(N)} \circledast G] + [f^{(N-1)}(0) \cdot G]$$
$$= [\mathbf{f}^{(N)} \circledast G] + [\tfrac{1}{C_N} \cdot G] .$$

Using the bilinearity of convolution, (19.2), we have that sum $\sum_{j=0}^{N} C_j y^{(j)}$ [which is LhS(25a)] equals

$$*: \quad \Big[\Big[\sum_{j=0}^{N} C_j f^{(j)}\Big] \circledast G\Big] + C_N \cdot [\tfrac{1}{C_N} \cdot G] .$$

Since $\sum_{j=0}^{N} C_j f^{(j)}$ is the zero-fnc, the convolution in $(*)$ is 0. Hence $(*)$ equals G, as requested. ♦

Gen soln to (25a). Recall that the general ZeroTar solution $Z()$ to $[q(\mathbf{D})](Z) = 0$ has N free parameters, $\alpha_1, \alpha_2, \ldots, \alpha_N \in \mathbb{C}$. Writing

$$\vec{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_N) ,$$

then, we denote the general *ZeroTar* soln as $Z_{\vec{\alpha}}(t)$. It follows that the sum

25c: $$Y_{\vec{\alpha}} := [f \circledast G] + Z_{\vec{\alpha}}$$

is the *general* GenTar-Soln to (25a) □

The following convolution-example will be done again using *Variation of Parameters* at (26.10).

Convol-GenTar Ex.1. We crave a particular soln, for $t{>}0$, to [#7P191,§4.6,**NSS9**] DE

25a†: $$y'' + 4y' + 4y = e^{-2t} \cdot \log(t) .$$

Define $\mathsf{L} := \mathbf{D}^2 + 4\mathbf{D} + 4\mathbf{I}$.

1st-step. Here, $N = 2$ and $\frac{1}{C_N} = \frac{1}{1} = 1$.
Aux.poly of L is $z^2 + 4z + 4 = [z - {-2}]^2$. Thus $f(x) := \alpha{\cdot}e^{-2x} + \beta{\cdot}xe^{-2x}$ satisfies $\mathsf{L}(f) = 0$.
Needing $f(0){=}0$ and $f'(0){=}1$ makes $\alpha{=}0$ and $\beta{=}1$. Hence $\boxed{f(x) = xe^{-2x}}$.

2nd-step. The (25a†)-target is $G(v) := e^{-2v}{\cdot}\log(v)$. Convolving, $[f \circledast G](t) \stackrel{\text{def}}{=} \int_0^t f(t-v){\cdot}G(v)\,dv$. The integrand is $[t-v]e^{-2[t-v]} \cdot e^{-2v}\log(v) \stackrel{\text{note}}{=} [t-v]e^{-2t} \cdot \log(v)$.

Thus $$[f \circledast G](t) = [[t{\cdot}\mathcal{A}] - \mathcal{B}] \cdot e^{-2t}, \quad \textit{where}$$
$$\mathcal{A} := \int_0^t \log(v)\,dv \quad \textit{and} \quad \mathcal{B} := \int_0^t v\log(v)\,dv .$$

IBParting, $\int^x \log = x[\log(x) - 1]$. Consequently,

$$\mathcal{A} = t[\log(t) - 1] - \lim_{s \searrow 0} s[\log(s) - 1] = t[\log(t) - 1] ,$$

since l'Hôpital's Thm shows $\lim_{s \searrow 0} s[\log(s) - 1]$ is zero.

Similarly, $\int^x v\log(v)\,dv = \frac{1}{4}[x^2[2\log(x) - 1]]$. So

$$\mathcal{B} = \int_0^t v\log(v)\,dv = \tfrac{1}{4}[t^2[2\log(t) - 1]] ,$$

again using l'Hôpital's. Hence $[t{\cdot}\mathcal{A}] - \mathcal{B}$ equals

$$\tfrac{1}{4}[t^2[4\log(t) - 4]] - \tfrac{1}{4}[t^2[2\log(t) - 1]] , \quad \textit{so}$$
$$y(t) = \tfrac{1}{4}t^2\big[2\log(t) - 3\big] \cdot e^{-2\cdot t}$$

is a particular soln to (25a†).

VoP, at (26.10), solves the same problem. Which method is easier?

Convol-GenTar Ex.2. We seek the gen-soln to

25a‡: $$3h'' - 4h' + h = \exp .$$

So $q(z) := 3z^2 - 4z + 1 = [z-1]\cdot[3z-1]$ is the aux-poly of our DiffOp, $\mathsf{L} := 3\mathbf{D}^2 - 4\mathbf{D} + \mathbf{I}$.

Applying 1st-step. Here, $N = 2$ and $\frac{1}{C_N} = \frac{1}{3}$.

The gen-soln to the ZeroTar DE $\big[q(\mathbf{D})\big](f) = 0$ is $f(x) := \alpha e^x + \beta e^{x/3}$. Solving for α,β so that $f(0) = 0$ and $f'(0) = \frac{1}{3}$ gives $f(x) = \frac{1}{2}\cdot[e^x - e^{x/3}]$. I.e

$$f = \tfrac{1}{2}\cdot[\exp - \Phi],$$

where $\Phi(x) := e^{x/3}$.

Applying 2nd-step. The target in (25a‡) is exp. The 2nd-step has us compute $h := f \circledast \exp$. Since convolution is bilinear,

$$h = \tfrac{1}{2}\cdot\Big[[\exp\circledast\exp] - [\Phi\circledast\exp]\Big].$$

By (21a), our $[\exp\circledast\exp](t) = t\cdot e^t$. And courtesy (21b),

$$[\Phi\circledast\exp](t) = \frac{e^t - e^{\frac{1}{3}t}}{1-\frac{1}{3}} = \frac{3}{2}\cdot\Big[e^t - e^{\frac{1}{3}t}\Big].$$

Consequently, our *General-target Soln* is

25c‡: $$H_{\alpha_1,\alpha_2}(t) = \tfrac{1}{2}te^t + \alpha_1 e^t + \alpha_2 e^{\frac{1}{3}t}.$$

A subtlety: We never needed to compute $[\Phi\circledast\exp]$, once we noticed from (21b) that $[\Phi\circledast\exp]$ is a linear-comb of Φ and exp. For the ZeroTar solns are all such lin-combs, so computing this specific one is irrelevant.

Precaution is called the Mother of Wisdom;
the father was never known.
That should prove to you, at at glance,
that even Precaution once took a chance.

—*Paul von der Porten*, *translated from the German by his son*, *Arnold von der Porten.*

Variation of Parameters [NSS9: §4.6 & §2.4., ex.#30]

[This section assumes knowledge of matrix multiplication, and the determinant of a square matrix.]

26.1: Cramer's "Rule" Thm. *Consider matrices* H *and* T, *and invertible matrix* M, *related by matrix-eqn*

$$\underbrace{\mathsf{M}}_{N\times N}\cdot\underbrace{\mathsf{H}}_{N\times 1} \;=\; \underbrace{\mathsf{T}}_{N\times 1}.$$

Here, "Multipler" M *and "Target"* T *are known, but "Huh?"* H *is unknown. Let* $\mathsf{M}_{\mathsf{T},r}$ *be the* $N{\times}N$ *matrix* M *except that its* r^{th}*-column has been replaced by column-vector* T. *With* h_r *the entry in the* r^{th}*-row of* H, *then*

$$h_r \;=\; \operatorname{Det}(\mathsf{M}_{\mathsf{T},r})/\operatorname{Det}(\mathsf{M})\,. \qquad \diamondsuit$$

Proof. The Determinant fnc is multiplicative, etc. ♦

A list $\vec{\varphi} := (\varphi_0, \varphi_1, \ldots, \varphi_{N-1})$ of sufficiently differentiable fncs engenders its ***Wronskian Matrix***

$$\mathsf{WM}(\vec{\varphi}) := \begin{bmatrix} \varphi_0 & \varphi_1 & \cdots & \varphi_{N-1} \\ \varphi_0' & \varphi_1' & \cdots & \varphi_{N-1}' \\ \varphi_0'' & \varphi_1'' & \cdots & \varphi_{N-1}'' \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_0^{(N-1)} & \varphi_1^{(N-1)} & \cdots & \varphi_{N-1}^{(N-1)} \end{bmatrix},$$

also written as $\mathsf{WM}(\varphi_0, \ldots, \varphi_{N-1})$. Its determinant,

$$\mathcal{W}(\varphi_0, \ldots, \varphi_{N-1}) \;:=\; \mathcal{W}(\vec{\varphi}) \;:=\; \operatorname{Det}(\mathsf{WM}(\vec{\varphi}))\,,$$

is called the "***Wronskian*** of $\vec{\varphi}$".

26.2: Wronskian L.I. Thm. *If* $\vec{\varphi} := (\varphi_0, \varphi_1, \ldots, \varphi_{N-1})$ *is a linearly-dependent list of functions, then* $\mathcal{W}(\vec{\varphi})$ *is the zero-function.*

Conversely, when each φ_j *is analytic* [is a power-series fnc]*: If* $\mathcal{W}(\vec{\varphi})$ *is the zero-function, then* $\vec{\varphi}$ *is linearly-dependent.* ♢

VoP algorithm [Variation of Parameters]

Step VoP0. Consider target fnc $G()$ and monic complex-polynomial

$$q(z) \;:=\; z^N + C_{N-1}z^{N-1} + \ldots + C_1 z^1 + C_0 z^0\,.$$

The polynomial determines a differential operator $\boxed{\mathsf{L} := q(\mathbf{D})}$. We seek the general solution, y, to $\mathsf{L}(y) = G$, i.e,

26.3: $y^{(N)} + C_{N-1}y^{(N-1)} + \ldots + C_1 y' + C_0 y \;=\; G\,.$

Step VoP1. Use CCLDE to find a linearly-independent list $\vec{\mathbf{Y}} := (Y_0, \ldots, Y_{N-1})$ of fncs, with each Y_j satisfying $\mathsf{L}(Y_j) = 0$.

We seek a list $\vec{\mathbf{f}} := (f_0, \ldots, f_{N-1})$ of fncs, so that this sum-function

26.4: $$s \;:=\; \sum_{j=0}^{N-1} f_j \cdot Y_j$$

satisfies (26.3); that is, that $\mathsf{L}(s) = G$.

VoP2. Let $h_j := f_j'$. Define column-vectors

26.5: $$\mathsf{H} := \begin{bmatrix} h_0 \\ \vdots \\ h_{N-2} \\ h_{N-1} \end{bmatrix} \quad\text{and}\quad \mathsf{T} := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ G \end{bmatrix}.$$

Compute the Wronskian matrix $\mathsf{M} := \mathsf{WM}(\vec{\mathbf{Y}})$. Then H satisfies

†: $$\underbrace{\mathsf{M}}_{N\times N}\cdot\underbrace{\mathsf{H}}_{N\times 1} \;=\; \underbrace{\mathsf{T}}_{N\times 1}.$$

Solve for each h_j, either via computing the inverse-matrix of M, or via Cramer's Rule (theorem, actually).

VoP3. Anti-differentiate to compute each function $f_j := \int h_j$. Then, parametrized by a list of numbers $\vec{\alpha} := (\alpha_0, \alpha_1, \ldots, \alpha_{N-1})$, the *general soln* to (26.3) is

26.6: $$y_{\vec{\alpha}} \;:=\; \Big[\sum_{j=0}^{N-1} \alpha_j Y_j\Big] + \Big[\sum_{j=0}^{N-1} f_j \cdot Y_j\Big].$$

Why does this nifty VoP algorithm work? Matrix-eqn (†) says, for $k = 0, 1, \ldots, N{-}2$, that

‡(k): $$\sum_{j=0}^{N-1} h_j \cdot Y_j^{(k)} \;=\; 0\,.$$

Differentiating (26.4) says that s' equals

$$\sum_{j=0}^{N-1} [f_j' Y_j + f_j Y_j'] \overset{\text{note}}{=} \Big[\sum_{j=0}^{N-1} h_j Y_j\Big] + \Big[\sum_{j=0}^{N-1} f_j Y_j'\Big].$$

By (‡(0)), then,

$$s' = \sum_{j=0}^{N-1} f_j Y_j' .$$

Differentiating again, then using (‡(1)), shows that

$$s'' = \sum_{j=0}^{N-1} f_j Y_j'' .$$

Continuing, we conclude, for $k = 1, 2, \ldots, N-1$, that

$$*: \quad s^{(k)} = \sum_{j=0}^{N-1} f_j Y_j^{(k)} .$$

Differentiating one last time produces

$$**: \quad s^{(N)} = \underbrace{\Big[\sum_{j=0}^{N-1} h_j Y_j^{(N-1)}\Big]}_{=:\ \text{Bob}} + \Big[\sum_{j=0}^{N-1} f_j Y_j^{(N)}\Big] .$$

Eqns (26.4), (∗) and (∗∗), together, imply that

$$\mathsf{L}(s) = \text{Bob} + \Big[\sum_{j=0}^{N-1} f_j \cdot \mathsf{L}(Y_j)\Big] .$$

But each $\mathsf{L}(Y_j) = 0$. Our end result is that

$$26.7: \quad \mathsf{L}(s) = \sum_{j=0}^{N-1} h_j Y_j^{(N-1)} .$$

And $\mathsf{L}(s) \stackrel{\text{want}}{=\!=} G$. Hence we need to require that H satisfies $\sum_{j=0}^{N-1} h_j Y_j^{(N-1)} = G$. And this is precisely what the bottom row of matrix-eqn (†) says.

The Upshot. This method *indeed* computes an s with $\mathsf{L}(s) = G$ *if* there is a column-vector H fulfilling (†). Happily, our Wronskian L.I. Thm (26.2) guarantees that M is invertible, since we chose $\vec{\mathbf{Y}}$ to be linearly-independent. So define $\boxed{\mathsf{H} := \mathsf{M}^{-1}\mathsf{T}}$. ♦

26.8: *VoP case $N{=}2$.* Here, our matrix eqn is

$$\underbrace{\begin{bmatrix} Y_0 & Y_1 \\ Y_0' & Y_1' \end{bmatrix}}_{\mathsf{M}} \cdot \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} = \begin{bmatrix} 0 \\ G \end{bmatrix} .$$

So $D := \operatorname{Det}(\mathsf{M}) = [Y_0 Y_1'] - [Y_0' Y_1]$. Hence

$$h_0 = -Y_1 \cdot \tfrac{G}{D} \quad \text{and} \quad h_1 = Y_0 \cdot \tfrac{G}{D} . \quad \text{Thus}$$

$$\begin{aligned} y_{\alpha,\beta} &= [\alpha + \textstyle\int h_0] Y_0 + [\beta + \textstyle\int h_1] Y_1 \\ &= [\alpha + f_0] Y_0 + [\beta + f_1] Y_1 \\ &\stackrel{\text{or}}{=} [\alpha Y_0 + \beta Y_1] + [f_0 Y_0 + f_1 Y_1] \end{aligned}$$

is our general soln to (26.3). □

26.9: General VoP Alg. *When the DE is "not monic", i.e*

$$26.3^*: \quad \mathrm{C}_N y^{(N)} + \mathrm{C}_{N-1} y^{(N-1)} + \ldots + \mathrm{C}_1 y' + \mathrm{C}_0 y = G ,$$

then steps VoP1,2,3 *remain, except that the target col-vec becomes*

$$26.5^*: \quad \mathsf{T} := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ G/\mathrm{C}_N \end{bmatrix} .$$

The algorithm persists if the C_j coefficients are allowed to be <u>*functions*</u> *of the independent variable. The only step that get harder is* VoP1 [*finding fncs sent to zero by the Diff-Op*] *since* CCLDE *no longer applies.* ♢

CC-VoP Example 1. DE [#7^P 191, §4.6, **NSS9**] is

$$26.10: \quad y'' + 4y' + 4y = \mathrm{e}^{-2t} \cdot \log(t) ,$$

for $t{>}0$. Define expressions

$$\mathcal{L} := \log(t) \quad \text{and} \quad \mathcal{R} := \mathrm{e}^{-2t} . \quad \text{Note } \mathcal{R}' = -2\mathcal{R} .$$

Our target fnc is $G := \mathcal{R} \cdot \mathcal{L}$.

VoP1. The Op's aux.poly is $z^2 + 4z + 4 = [z - -2]^2$. So

$$Y_0 := \mathcal{R} \quad \text{and} \quad Y_1 := t\mathcal{R} .$$

is an L.I. pair of fncs annihilated by the DiffOp.

VoP2. Differentiating w.r.t t,

$$Y_0' = -2\mathcal{R} \quad \text{and} \quad \begin{aligned} Y_1' &= 1 \cdot \mathcal{R} + t \cdot [-2\mathcal{R}] \\ &= [1 - 2t]\mathcal{R} . \end{aligned}$$

So the Wronskian-determinant $D := \mathcal{W}(Y_0, Y_1)$ is

$$D = \mathcal{R} \cdot [1 - 2t]\mathcal{R} - t\mathcal{R} \cdot [-2\mathcal{R}] \stackrel{\text{note}}{=\!=} \mathcal{R}^2 .$$

Using the convenient (26.8),

$$h_0 = -\tfrac{1}{D} Y_1 G = -\mathcal{R}^{-2} \cdot t\mathcal{R} \cdot \mathcal{R}\mathcal{L} \stackrel{\text{note}}{=\!=} -t\mathcal{L} , \quad \text{and}$$

$$h_1 = \tfrac{1}{D} Y_0 G = \mathcal{R}^{-2} \cdot \mathcal{R} \cdot \mathcal{R}\mathcal{L} \stackrel{\text{note}}{=\!=} \mathcal{L} .$$

VoP3. Computing anti-derivatives,

$$f_0 = \int [-t \cdot \mathcal{L}]\,\mathrm{d}t \;=\; \tfrac{1}{4}[1 - 2\mathcal{L}] \cdot t^2 \quad \text{and}$$
$$f_1 = \int \mathcal{L}\,\mathrm{d}t \;=\; [\mathcal{L} - 1] \cdot t\,.$$

So $\underline{a}$ fnc s sent to G by $\mathsf{L} \coloneqq \mathbf{D}^2 + 4\mathbf{D} + 4\mathbf{I}$ is

$$\begin{aligned} f_0 Y_0 + f_1 Y_1 &= f_0\,\mathcal{R} + f_1\,t\mathcal{R} \\ &= \Big[\tfrac{1}{4}[1 - 2\mathcal{L}] + [\mathcal{L} - 1]\Big] \cdot t^2\mathcal{R} \\ &= \tfrac{1}{4}[2\mathcal{L} - 3]t^2\mathcal{R} \;=\; \tfrac{1}{4}[2\log(t) - 3]t^2\mathrm{e}^{-2t}. \end{aligned}$$

The gen. $y_{\alpha,\beta} \coloneqq \alpha Y_0 + \beta Y_1 = [\alpha + \beta t]\mathcal{R}$ is annihilated by L. Hence, the gen. $s_{\alpha,\beta}$ with $\mathsf{L}(s_{\alpha,\beta}) = G$ is

$$\begin{aligned} s_{\alpha,\beta} &= [\alpha + \beta t]\mathcal{R} + \big[\,\tfrac{1}{4}[2\mathcal{L} - 3] \cdot t^2\mathcal{R}\big] \\ &= \Big[[\alpha + \beta t] + \tfrac{1}{4}[2\log(t) - 3]\,t^2\Big] \cdot \mathrm{e}^{-2t}\,. \end{aligned}$$

Convol-GenTar, at (25a†), solves the Same Problem. *Which method is easier?*

A WONDERFUL BIRD IS THE PELICAN

His bill holds more than his belican.
He can take in his beak,
Enough food for a week,
But I'm damned if I see how the helican.

–Dixon Lanier Merritt

Equidimensional operators

Motivation. Here, we act on functions of t. Equidimensional operators are designed to annihilate a power of t; some $t^{\mathbf{r}}$, where $\mathbf{r}$ *need not* be an integer. Indeed, if we only consider values $t>0$, then we can allow $\mathbf{r}$ to be complex, recalling that $t^{\mathbf{r}} \stackrel{\text{def}}{=} \exp(\log(t)\cdot\mathbf{r})$. □

Defn. An "***equidimensional operator*** of order 2" [***EquiDim-Op***] has form

$$\mathsf{E}(y) \quad := \quad At^2y'' + Bty' + Cy$$

where $A_{\neq 0}, B, C \in \mathbb{C}$ and $y=y(t)$. [See §4.7 in **NSS9**, where such operators are called ***Cauchy-Euler operators*** as well as *equidimensional*.]

A ***Generalized EquiDim-Op*** [abbrev. ***Gen-EquiDim-Op***] has form

$$\mathsf{E}(y) \quad := \quad At^{\Lambda+2}y'' + Bt^{\Lambda+1}y' + Ct^{\Lambda}y$$

for some $\Lambda \in \mathbb{C}$.

For a number $\mathbf{r} \in \mathbb{C}$, observe that

27a:
$$\begin{aligned}\mathsf{E}(t^{\mathbf{r}}) &= At^{\Lambda+2}\cdot\mathbf{r}[\mathbf{r}-1]t^{\mathbf{r}-2} \\ &\quad + Bt^{\Lambda+1}\cdot\mathbf{r}t^{\mathbf{r}-1} + Ct^{\Lambda}\cdot t^{\mathbf{r}} \\ &= t^{\Lambda+\mathbf{r}}\cdot q(\mathbf{r}),\end{aligned}$$

$$\text{where} \quad q(z) := Az^2 + [B-A]z + C \qquad \square$$

is the "***characteristic polynomial*** of E".

The quadratic formula gives the roots, $\mathbf{r}_1$ and $\mathbf{r}_2$, of q. Hence E sends $t^{\mathbf{r}_1}$ and $t^{\mathbf{r}_2}$ to the zero-fnc. If $\operatorname{Discr}(q)=0$, i.e $\mathbf{r}_1=\mathbf{r}_2$, then we can apply the below Reduction-of-order method. This will give us a fnc $s()$ which is L.I of $t^{\mathbf{r}_1}$ s.t $\mathsf{E}(s)=0$.

Roo algorithm [Reduction of order]

Consider coefficient-*functions* $C_j = C_j(t)$, defining linear Diff-Op

$$\mathsf{L}(\varphi) := \varphi'' + C_1\varphi' + C_0\varphi.$$

Suppose we have a fnc Y, which is not identically-zero, satisfying $\mathsf{L}(Y)=0$.

Given a target fnc G, we seek a fnc s which is linearly-indep of Y, s.t $\mathsf{L}(s)=G$. This s will have for $Y\cdot f$ for an f we will compute. We start by computing $h := f'$ by means of FOLDE.

Step Roo1. Compute an anti-deriv $B_1 := \int C_1$, then let

$$M := Y^2\cdot \mathrm{e}^{B_1}.$$

Roo2. *If* G is identically-zero, then set

$$h := \frac{1}{M} \stackrel{\text{note}}{=} \frac{1}{Y^2}\cdot\mathrm{e}^{-B_1}.$$

Otherwise, define

$$h := \frac{1}{M}\cdot\int\frac{MG}{Y} \stackrel{\text{note}}{=} \frac{1}{M}\cdot\int[Y\cdot\mathrm{e}^{B_1}\cdot G].$$

Roo3. Compute an anti-derivative

$$f := \int h. \quad \text{Finally, define} \quad s := Y\cdot f.$$

Why does the Roo alg. work? We solve for a fnc f such that $s := Y\cdot f$ satisfies $\mathsf{L}(s)=G$. Let $h := f'$. Differentiating

$$\begin{aligned} s &= Y\cdot f \quad \textit{produces} \\ s' &= Y'f + Yf' \\ &\stackrel{\text{note}}{=} Y'f + Yh. \quad \textit{Thus} \\ s'' &= Y''f + Y'f' + Y'h + Yh' \\ &= Y''f + [Yh' + 2Y'h].\end{aligned}$$

Thus $\mathsf{L}(s) \stackrel{\text{def}}{=} s'' + C_1s' + C_0s$ equals

$$\begin{aligned}&\mathsf{L}(Y)\cdot f + [Yh' + 2Y'h] + C_1Yh \\ &\stackrel{\text{since } \mathsf{L}(Y)=0}{=\!=\!=} Yh' + [2Y' + C_1Y]h.\end{aligned}$$

Consequently, h satisfies $Yh' + [2Y' + C_1Y]h = G$. Dividing by Y yields FOLDE

27b:
$$h' + \overbrace{[[2\tfrac{Y'}{Y}] + C_1]}^{\text{FOLDE coeff-fnc}}\cdot h = \overbrace{\tfrac{G}{Y}}^{\text{FOLDE target-fnc}}.$$

Note $\frac{Y'}{Y} = [\log(Y)]'$, so $2\frac{Y'}{Y} = [2\log(Y)]' = [\log(Y^2)]'$. Thus the FOLDE anti-deriv of the coeff-fnc is

$$B := \int[2\tfrac{Y'}{Y} + C_1] \stackrel{\text{note}}{=} \log(Y^2) + B_1.$$

Hence the FOLDE multiplier-fnc is

$$M := Y^2\cdot\mathrm{e}^{B_1}.$$

The last FOLDE-step gives the two formulas in **Roo2**. ♦

> *Measure twice, cut once.* –**Proverb**

Equidim + Roo Example 1. For $t>0$, let's find the gen.soln $\varphi=\varphi(t)$ of DE

27c: $$t^2\varphi'' - 5t\varphi' + 9\varphi = 0.$$

Operator $\mathsf{E}(y) := t^2y'' - 5ty' + 9y$ is equidimensional. Its char-poly is, from (27a),

$$z^2 + [-5-1]z + 9 = z^2 - 6z + 9 = [z-3]^2.$$

Hence $Y(t) := t^3$ is sent to 0 by $\mathsf{E}()$. Checking:

$$\mathsf{E}(t^3) = t^2\cdot[3\cdot 2t] - 5t\cdot[3t^2] + 9\cdot[t^3]$$
$$= 3t^3\cdot[2 - 5 + 3] \overset{\text{note}}{=\!=} 0.$$

Roo1. We make a monic version of the operator by defining $\mathsf{L} := [1/t^2]\cdot\mathsf{E}$, i.e

$$\mathsf{L}(y) := y'' - \tfrac{5}{t}y' + \tfrac{9}{t^2}y.$$

With $C_1(t) := -\frac{5}{t}$, then

$$B_1(t) := \int^t C_1 = -5\cdot\log(t).$$

So $\exp(B_1(t))$ equals t^{-5}. Thus

$$M(t) := [t^3]^2\cdot t^{-5} = t.$$

Roo2. Our target fnc is the zero-fnc, so we simply compute
$$h(t) := 1/M(t) = 1/t.$$

Roo3. Antidifferentiating gives $f := \int h = \log$. Consequently, the theory tells us that

27d: $$s(t) := f(t)\cdot[Y(t)] \overset{\text{note}}{=\!=} \log(t)\cdot t^3$$

is sent to the zero-fnc by L [hence also by E], *and* is L.I of $Y(t)=t^3$. *Did you check?*

Checking: Let $\mathcal{G} := \log(t)$. Then

$$s = \mathcal{G}t^3. \quad \text{Thus}$$
$$s' = \tfrac{1}{t}t^3 + \mathcal{G}\cdot 3t^2 = [1 + 3\mathcal{G}]t^2, \quad \text{so}$$
$$s'' = \tfrac{3}{t}\cdot t^2 + [1+3\mathcal{G}]\cdot 2t = [5 + 6\mathcal{G}]t. \quad \textit{Summing}$$
$$9s = [0 + 9\mathcal{G}]\,t^3 \quad \text{with}$$
$$-5ts' = [-5 - 15\mathcal{G}]\,t^3 \quad \text{and with}$$
$$t^2s'' = [5 + 6\mathcal{G}]\,t^3$$

is the defn of $\mathsf{E}(s)$. The sum is indeed zero.

Roo Example 2. For $\varphi = \varphi(x)$, define operator

$$\mathsf{L}(\varphi) \quad := \quad \varphi'' - \tan(x)\varphi' - [1+\tan(x)^2]\varphi .$$

Given that $\mathsf{L}(\tan) = 0$, we seek the general solution $g = g(x)$ to

28a: $$\mathsf{L}(g) = 1 .$$

As tan() blows up at $\pm\frac{\pi}{2}$, we restrict to $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Note that cos() is is positive on $(-\frac{\pi}{2}, \frac{\pi}{2})$.

Sanity check. Define abbreviations

$$\mathbf{C} := \cos(x), \quad \mathbf{S} := \sin(x), \quad \mathbf{T} := \tan(x) \stackrel{\text{note}}{=\!=} \frac{\mathbf{S}}{\mathbf{C}} .$$

Let's *verify* that what we were given is true. Note

$$\mathbf{T}' = [1+\mathbf{T}^2], \quad \textit{hence} \quad \mathbf{T}'' = 2\mathbf{T}[1+\mathbf{T}^2] .$$

Thus $\mathsf{L}(\mathbf{T})$ equals

$$2\mathbf{T}[1+\mathbf{T}^2] - \mathbf{T}[1+\mathbf{T}^2] - [1+\mathbf{T}^2]\mathbf{T} ,$$

which is indeed zero. □

Gen. ZeroTar soln. To find a fnc $s = s(x)$ st. $\mathsf{L}(s) = 0$ *and* pair $\{\mathbf{T}, s\}$ is L.I (linearly indep), the Roo method has us compute a fnc f so that $s := \mathbf{T}\cdot f$ achieves these goals. This $f := \int h$ for an h that we now compute.

Computing h. Using Roo notation, $C_1 = -\mathbf{T}$ and $C_0 = -[1+\mathbf{T}^2]$. Note $B_1 := \int C_1 = \log\circ\mathbf{C}$. Consequently, $\mathsf{e}^{B_1} = \exp\circ\log\circ\mathbf{C} \stackrel{\text{note}}{=\!=} \mathbf{C}$. Our FOLDE multiplier is thus $M := \mathbf{T}^2 \cdot \mathsf{e}^{B_1} = \mathbf{T}^2\cdot\mathbf{C} = \mathbf{S}^2/\mathbf{C}$.

In the ZeroTar case, Roo says

$$h = 1/M = \mathbf{C}/\mathbf{S}^2 . \quad \textit{Thus}$$

$$f \stackrel{\text{def}}{=\!=} \int h = -1/\mathbf{S} .$$

Roo says to define $s := \mathbf{T}\cdot f \stackrel{\text{note}}{=\!=} -1/\mathbf{C}$. But since the *target is zero*, and L is *linear*, we may freely multiply by a non-zero constant. Hence, we shall define s as $s := 1/\mathbf{C}$.

CHECK: To verify that s is annihilated by $\mathsf{L}()$, note

$$s' = \frac{1}{\mathbf{C}}\mathbf{T} . \quad \textit{Thus,}$$

$$s'' = [\tfrac{1}{\mathbf{C}}\mathbf{T}]\cdot\mathbf{T} + \tfrac{1}{\mathbf{C}}[1+\mathbf{T}^2] = \tfrac{1}{\mathbf{C}}[1+2\mathbf{T}^2]. \quad \textit{So,}$$

$$\mathsf{L}(s) = \frac{1}{\mathbf{C}}\cdot\Big[[1+2\mathbf{T}^2] - \mathbf{T}\cdot\mathbf{T} - [1+\mathbf{T}^2]\cdot 1\Big] \stackrel{\text{note}}{=\!=} 0,$$

as predicted by the theory. ♦

Remark. To solve $\mathsf{L}(g) = 1$ we could start with *either* ZeroTar soln; $\mathbf{T}$ or $\frac{1}{\mathbf{C}}$. But since we have *already computed* the multiplier-fnc for $\mathbf{T}$, we will use $\mathbf{T}$. □

Solving $\mathsf{L}(g) = 1$. Recall $M = \mathbf{T}^2\cdot\mathbf{C}$, when using ZeroTar $\mathbf{T}$. Roo asserts that our h is

$$h := \frac{1}{M}\cdot\int\frac{MG}{\mathbf{T}} ,$$

where, now, $G \equiv 1$ is our target function. The integrand is $\mathbf{T}\cdot\mathsf{e}^{B_1}\cdot G = \mathbf{T}\cdot\mathbf{C}\cdot 1 = \mathbf{S}$. Hence

$$h = \frac{1}{M}\cdot\int\mathbf{S} = \frac{1}{\mathbf{T}^2\,\mathbf{C}}\cdot[-\mathbf{C}] = \frac{-1}{\mathbf{T}^2}. \quad \textit{Thus,}$$

$$f(x) \stackrel{\text{def}}{=\!=} \int^x h = \frac{1}{\mathbf{T}} + x . \quad \textit{Consequently,}$$

$$g(x) = \mathbf{T}\cdot f(x) = 1 + [x\mathbf{T}] .$$

It is tedious, but easy, to verify $\mathsf{L}(1 + x\mathbf{T}) = 1$. So

28b: $$g_{\alpha,\beta}(x) := [1 + x\tan(x)] + \alpha\tan(x) + \frac{\beta}{\cos(x)}$$

is the gen.soln to $\mathsf{L}(g_{\alpha,\beta}) = 1$. ♦

Other methods. We solved DE (28a) via Roo + Roo. Alternatively, we could have used Roo + VoP or algorithm Roo + Convol-GenTar. □

Tarantulas tarantulas
Everybody loves tarantulas
If there's just fuzz where your hamster was
It's probably because of tarantulas

–*chorus of "The Tarantula Song"* –*Bryant Oden*

Stopped at a traffic light, the car in front has vanity plate ML8ML8 . *What color is the car?* 7i

Operators

We already know operators $\mathbf{D}$ and $\mathbf{I}{=}\mathbf{D}^0$. Use $\mathbf{0}$ for the ***zero-operator***. I.e, $\mathbf{0}(y) = 0$ [the zero-fnc] for *every* fnc y.

Translation. Use $\mathbf{T}$ for the family of ***translation operators***. For a *number* $\alpha \in \mathbb{C}$, operator $\mathbf{T}_\alpha$ acts on an arbitrary fnc φ to produce a new function, which is φ but translated [to the "right"] by α. E.g,

$$\mathbf{T}_5(\varphi) \quad = \quad [t \mapsto \varphi(t-5)] .$$

[So $\mathbf{T}_0 = \mathbf{I}$.] For instance, we know that cos() and sin() are translates of each other. Specifically

$$\mathbf{T}_{\pi/2}(\cos) \;=\; \sin \quad \text{and} \quad \mathbf{T}_{-\pi/2}(\sin) \;=\; \cos .$$

A [complex] number β is **"**a ***period*** of f**"** if $\mathbf{T}_\beta(f) = f$. E.g, $\mathbf{T}_{2\pi}(\cos) = \cos$. And $\mathbf{T}_{2\pi\mathbf{i}}(\exp) = \exp$.

Multiply-operator. Use $\mathbf{M}$ for the family of ***multiply operators***. So $\mathbf{M}_5$ multiplies its argument by [the constant fnc] 5, e.g $\mathbf{M}_5(y) = 5y$, i.e $\mathbf{M}_5 = 5\mathbf{I}$. More generally, for a *function* f, let $\mathbf{M}_f(y) := f{\cdot}y$. That is

$$[\mathbf{M}_f(y)](t) \;=\; f(t){\cdot}y(t) \stackrel{\text{abbrev.}}{=\!=} f(t){\cdot}y .$$

By slight abuse of notation, we can also use an *expression* as a subscript, e.g, $\mathbf{M}_{t^2}(y)$ means $t^2 y$; well, actually, the *function* $[t \mapsto t^2 y(t)]$.

29.1**:** Lemma. *Easily,* $\mathbf{M}_0 = \mathbf{0}$ *and* $\mathbf{M}_1 = \mathbf{I} = \mathbf{T}_0$. *Also:*

i: *Each* $\mathbf{T}_\alpha$ *is invertible, and* $[\mathbf{T}_\alpha]^{-1} = \mathbf{T}_{-\alpha}$.

ii: *When* f *is no-where zero, then* $\mathbf{M}_f$ *is invertible, with inverse* $\mathbf{M}_{1/f}$. ◊

Commutation relations. Boldface symbols

$$\mathbf{D},\ \mathbf{I},\ \mathbf{0},\ \mathbf{T}_?\ \text{and}\ \mathbf{M}_?$$

denote operators with fixed meanings. We'll use san-serif letters $\mathsf{L}, \mathsf{P}, \mathsf{Q}, \mathsf{U}, \mathsf{V}$ for *operator-variables*; variables that we can assign operators to. Make the convention that, e.g, VP means $\mathsf{V} \circ \mathsf{P}$, and V^3 means $\mathsf{V} \circ \mathsf{V} \circ \mathsf{V}$. Hence $\mathsf{V}^0 = \mathbf{I}$.

Use "$\leftrightarrows$" to mean 'commutes with'. So $\mathsf{U} \leftrightarrows \mathsf{V}$ means that $\mathsf{UV} = \mathsf{VU}$.

29.2**:** Op-commutation lemma. *Here* $\alpha,\beta \in \mathbb{C}$, *and* f,g *are functions.*

a: *Translation-ops are linear and commute with each other. Indeed,* $\mathbf{T}_\beta \mathbf{T}_\alpha = \mathbf{T}_{\beta+\alpha} = \mathbf{T}_\alpha \mathbf{T}_\beta$.

b: *Multiply-ops are linear and commute with each other. Specifically,* $\mathbf{M}_f \mathbf{M}_g = \mathbf{M}_{f\cdot g} = \mathbf{M}_g \mathbf{M}_f$.

c: *Each translation-op commutes with* $\mathbf{D}$.

d: *Operator* $\mathbf{M}_g$ *commutes with* $\mathbf{D}$ IFF g *is constant. The general commutation relation is*

$$\begin{aligned} \mathbf{D}\,\mathbf{M}_g &= \mathbf{M}_{g'} + [\mathbf{M}_g\,\mathbf{D}], \quad \textit{E.g,} \\ \mathbf{D}\,\mathbf{M}_t &= \mathbf{I} + [\mathbf{M}_t\,\mathbf{D}]. \end{aligned}$$

e: *Operator* $\mathbf{M}_f$ *commutes with* $\mathbf{T}_\beta$ IFF β *is a period of* f. *The commutation relation* [*written with composition symbol* $\circ$, *for clarity*] *is*

$$\mathbf{T}_\beta \circ \mathbf{M}_f \quad = \quad \mathbf{M}_{\mathbf{T}_\beta(f)} \circ \mathbf{T}_\beta . \qquad \Diamond$$

Proof of (c). Exercise. Use the Chain rule. ♦

Pf of (d). Well, $\mathbf{D}\mathbf{M}_g(y) = \mathbf{D}(g{\cdot}y) = g'y + gy'$, which equals $\mathbf{M}_{g'}(y) + [\mathbf{M}_g\mathbf{D}](y)$, i.e, $\Big[\mathbf{M}_{g'} + [\mathbf{M}_g\,\mathbf{D}]\Big](y)$. ♦

Pf of (e). Let $f_\beta := \mathbf{T}_\beta(f)$ and, for y an arbitrary fnc, let $y_\beta := \mathbf{T}_\beta(y)$. So $\mathbf{M}_{\mathbf{T}_\beta(f)} = \mathbf{M}_{f_\beta}$. Thus

$$*: \quad \mathbf{M}_{f_\beta}\mathbf{T}_\beta(y) \;=\; f_\beta{\cdot}y_\beta \;=\; \mathbf{T}_\beta(f{\cdot}y) \;=\; \mathbf{T}_\beta\mathbf{M}_f(y),$$

yielding the stated commutation relation.

Now, if $\mathbf{M}_\beta \leftrightarrows \mathbf{T}_f$ then $\mathbf{M}_f\mathbf{T}_\beta = \mathbf{T}_\beta\mathbf{M}_f = \mathbf{M}_{f_\beta}\mathbf{T}_\beta$, by ($*$). Evaluating at the constant function 1 shows that $\mathbf{M}_{f_\beta}(1) = \mathbf{M}_f(1)$. Consequently $f_\beta = f$. ♦

Example. *Numerical* expressions can be simplified [e.g $7+1$ equals 8], as can *fnc* expresions [e.g $\cos^2+\sin^2$ equals the constant-fnc $\mathbf{1}^2$], and so too can *operator* expressions. For example, the above lemma allows this

$$\mathbf{M}_5\mathbf{D}\mathbf{M}_{\sin}\mathbf{D} \overset{\text{by (29.2d)}}{=\!=\!=} \mathbf{M}_5\big[\mathbf{M}_{\cos} + \mathbf{M}_{\sin}\mathbf{D}\big]\mathbf{D}$$

$$\overset{\text{by (29.2b)}}{=\!=\!=} \mathbf{M}_{5\cos}\mathbf{D} + \mathbf{M}_{5\sin}\mathbf{D}^2\,.$$

Another: Note that

$$\mathbf{T}_{\boldsymbol{\pi}/2}\,\mathbf{M}_{\cos} \overset{\text{by (e)}}{=\!=} \mathbf{M}_{\mathbf{T}_{\boldsymbol{\pi}/2}(\cos)}\,\mathbf{T}_{\boldsymbol{\pi}/2} = \mathbf{M}_{\sin}\,\mathbf{T}_{\boldsymbol{\pi}/2}\,.$$

Hence

$$\mathbf{T}_{\boldsymbol{\pi}/2}\,\mathbf{M}_{\cos}\,\mathbf{T}_{3\boldsymbol{\pi}/2} = \mathbf{M}_{\sin}\,\mathbf{T}_{2\boldsymbol{\pi}}\,.$$

□

Matrix exponential

Fix posint N and let MAT denote the set of $N\times N$ matrices. Use $\mathbf{0},\mathbf{I} \in$ MAT for the zero-matrix and identity-matrix. For $\mathsf{M} \in$ MAT, define

$$*\!: \quad \exp(\mathsf{M}) \;:=\; \mathsf{e}^{\mathsf{M}} \;:=\; \sum_{k=0}^{\infty}\Big[\tfrac{1}{k!}\cdot \mathsf{M}^k\Big]\,.$$

30.1: *MiniChallenge: MatrixExp by hand.* Fix an $\boldsymbol{\alpha} \in \mathbb{C}$ and set $\mathsf{S} := \begin{bmatrix}\boldsymbol{\alpha} & \boldsymbol{\alpha}\\ 0 & 0\end{bmatrix}$. Compute e^{S} and $\mathsf{e}^{t\mathsf{S}}$. □

Soln. Let's do this for $\boldsymbol{\alpha} := 5$; we'll see the pattern.

Always, S^0 is the identity matrix $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$. And for $k \in \mathbb{Z}_+$, easily

$$\mathsf{S}^k \;=\; \begin{bmatrix}5^k & 5^k\\ 0 & 0\end{bmatrix}.$$

Writing S^0 in the same pattern, then,

$$\mathsf{S}^0 \;=\; \begin{bmatrix}5^0 & 5^0\\ 0 & 0\end{bmatrix} + \mathsf{C}\,, \qquad \text{where } \mathsf{C} := \begin{bmatrix}0 & -1\\ 0 & 1\end{bmatrix}.$$

Applying defn $(*)$, our $\mathsf{e}^{t\mathsf{S}}$ equals

$$\tfrac{1}{0!}\cdot t^0\cdot\mathsf{C} \;+\; \sum_{k=0}^{\infty}\tfrac{1}{k!}\cdot t^k \cdot \begin{bmatrix}5^k & 5^k\\ 0 & 0\end{bmatrix}$$

$$=\;\mathsf{C} \;+\; \begin{bmatrix}\sum_{k=0}^{\infty}\tfrac{1}{k!}\cdot t^k 5^k & \sum_{k=0}^{\infty}\tfrac{1}{k!}\cdot t^k 5^k\\ 0 & 0\end{bmatrix}.$$

This $\sum_{k=0}^{\infty}\tfrac{1}{k!}t^k 5^k$ is just the Taylor series of e^{5t}, so

$$\mathsf{e}^{t\mathsf{S}} \;=\; \begin{bmatrix}0 & -1\\ 0 & 1\end{bmatrix} + \begin{bmatrix}\mathsf{e}^{5t} & \mathsf{e}^{5t}\\ 0 & 0\end{bmatrix} \;\stackrel{\text{note}}{=\!=}\; \left[\begin{array}{c|c}\mathsf{e}^{5t} & \mathsf{e}^{5t}-1\\ 0 & 1\end{array}\right].$$

Nothing was special about the complex number 5, so for our original S we conclude that

$$\textbf{30.2:}\quad \mathsf{e}^{t\mathsf{S}} \;=\; \exp\Big(t\cdot\begin{bmatrix}\boldsymbol{\alpha} & \boldsymbol{\alpha}\\ 0 & 0\end{bmatrix}\Big) \;=\; \begin{bmatrix}\mathsf{e}^{\boldsymbol{\alpha}t} & \mathsf{e}^{\boldsymbol{\alpha}t}-1\\ 0 & 1\end{bmatrix}.$$

Plugging in $t{=}1$ gives

$$\textbf{30.3:}\quad \mathsf{e}^{\mathsf{S}} \;=\; \exp\Big(\begin{bmatrix}\boldsymbol{\alpha} & \boldsymbol{\alpha}\\ 0 & 0\end{bmatrix}\Big) \;=\; \begin{bmatrix}\mathsf{e}^{\boldsymbol{\alpha}} & \mathsf{e}^{\boldsymbol{\alpha}}-1\\ 0 & 1\end{bmatrix}.$$

By the way, at $t{=}0$, note that (30.2) is the identity matrix. *Coincidence? Space aliens? I think not!* ♦

Defn. An $N\times N$ matrix M is ***nilpotent*** if $\exists k \in \mathbb{Z}_+$ such that $\boxed{\mathsf{M}^k = \mathbf{0}_{N\times N}}$. The *smallest* such k is the "***nilpotency degree*** of M", written $\text{NilDeg}(\mathsf{M})$. [Thus "$\text{NilDeg}(\mathsf{M}) = \infty$" means M is *not* nilpotent.] Always:

The nilpotency degree of a nilpotent $N\times N$ matrix is $\leq N$.

Matrices $\mathsf{A},\mathsf{B} \in$ MAT are ***similar***[♡14] [to each other] if *there exists*[♡14] an invertible $\mathsf{U} \in$ MAT such that

$$\mathsf{B} \;=\; \mathsf{U}\mathsf{A}\mathsf{U}^{-1}\,. \quad \text{Write this relation as} \quad \mathsf{A} \sim \mathsf{B}\,.$$

Easily, relation $\sim$ is an equivalence relation.

This A is ***diagonalizable*** if A is similar to *some* diagonal matrix.

Read $\mathsf{A} \leftrightarrows \mathsf{B}$ as "A commutes with B" i.e, $\mathsf{AB} = \mathsf{BA}$. □

31: MatExp theorem. *Series $(*)$ always converges. Moreover, for scalars α,β and $\mathsf{A},\mathsf{B},\mathsf{R},\mathsf{D} \in$ MAT:*

a: *Exp() of a diagonal matrix* $\mathsf{D} := \begin{bmatrix}\alpha_1 & & \\ & \ddots & \\ & & \alpha_N\end{bmatrix}$ *yields diagonal matrix*

$$\mathsf{e}^{\mathsf{D}} = \begin{bmatrix}\mathsf{e}^{\alpha_1} & & \\ & \ddots & \\ & & \mathsf{e}^{\alpha_N}\end{bmatrix}, \text{ so } \quad \mathsf{e}^{t\mathsf{D}} = \begin{bmatrix}\mathsf{e}^{\alpha_1 t} & & \\ & \ddots & \\ & & \mathsf{e}^{\alpha_N t}\end{bmatrix}.$$

Thus $\mathsf{e}^{\mathbf{0}} = \mathbf{I}$.

b: *If matrices* $\mathsf{A} \leftrightarrows \mathsf{B}$, *then* $\mathsf{e}^{\mathsf{A}+\mathsf{B}} = \mathsf{e}^{\mathsf{A}}\cdot\mathsf{e}^{\mathsf{B}}$. *Hence, every* e^{R} *is invertible, and* $\big[\mathsf{e}^{\mathsf{R}}\big]^{-1} = \mathsf{e}^{-\mathsf{R}}$. *Also,* $\mathsf{e}^{[\alpha+\beta]\mathsf{R}} = \mathsf{e}^{\alpha\mathsf{R}}\cdot\mathsf{e}^{\beta\mathsf{R}}$.

c: *For R arbitrary and U invertible, let* $\mathsf{D} := \mathsf{U}^{-1}\mathsf{R}\mathsf{U}$; *so* $\mathsf{R} := \mathsf{U}\mathsf{D}\mathsf{U}^{-1}$. *Then* $\boxed{\mathsf{e}^{\mathsf{U}\mathsf{D}\mathsf{U}^{-1}} = \mathsf{U}\mathsf{e}^{\mathsf{D}}\mathsf{U}^{-1}}$. *I.e,*

[Conjugation by U] *commutes-with* exp().

From above, $t\mathsf{R} = \mathsf{U}\cdot t\mathsf{D}\cdot\mathsf{U}^{-1}$, *since scalars commute with matrices, and thus*

$$\mathsf{e}^{t\mathsf{R}} \;=\; \mathsf{U}\cdot\mathsf{e}^{t\mathsf{D}}\cdot\mathsf{U}^{-1}\,.$$

d: *Function* $[t \mapsto \mathsf{e}^{t\mathsf{R}}]$ *is differentiable, and*

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\mathsf{e}^{t\mathsf{R}} \;=\; \mathsf{R}\cdot\mathsf{e}^{t\mathsf{R}} \;=\; \mathsf{e}^{t\mathsf{R}}\cdot\mathsf{R}\,.$$ ◇

[♡14] We also say "*A and B are conjugate to each other*", or "*matrix U conjugates A to B.*" In general, U is *not* unique; there could be an invertible $\mathsf{W}\neq\mathsf{U}$ s.t $\mathsf{W}\mathsf{A}\mathsf{W}^{-1} = \mathsf{B} = \mathsf{U}\mathsf{A}\mathsf{U}^{-1}$.

32.1: *MiniChallenge: CEX to* $e^{A+B} = e^A e^B$.
Find 2×2 matrices A and B which form a counterexample (abbrev. CEX) to assertion $e^{A+B} = e^A e^B$. □

Soln. MatExp (31b) tells us to search among non-commuting pairs; that is, $AB \neq BA$. About the simplest non-commuting pair there is, is

32.2: $$A := \begin{bmatrix}1&0\\0&0\end{bmatrix} \quad\text{and}\quad B := \begin{bmatrix}0&1\\0&0\end{bmatrix}.$$

Is this pair a CEX ?! (*This is so exciting!*)

Since A is a diagonal matrix, our (31a) says

$$e^A = \begin{bmatrix}e^1&0\\0&e^0\end{bmatrix} = \begin{bmatrix}e&0\\0&1\end{bmatrix}.$$

Our B has nilpotency-degree 2 [i.e $B^2 = \mathbf{0}_{2\times2}$], so

$$e^B = \tfrac{1}{0!}\mathbf{I} + \tfrac{1}{1!}B = \mathbf{I} + B = \begin{bmatrix}1&1\\0&1\end{bmatrix}.$$

Before even computing e^{A+B}, note that

32.3: $$e^A \cdot e^B = \begin{bmatrix}e&e\\0&1\end{bmatrix} \neq \begin{bmatrix}e&1\\0&1\end{bmatrix} = e^B \cdot e^A.$$

Since A+B *does* equal B+A, this implies that –in one order or the other– we indeed have a CEX.

To find out which, we compute e^S, where the sum

$$S := A + B \overset{\text{note}}{=\!=} \begin{bmatrix}1&1\\0&0\end{bmatrix}.$$

Our previous work, (30.3), says that exponential

32.4: $$e^S = \begin{bmatrix}e^1&e^1-1\\0&1\end{bmatrix} \overset{\text{note}}{=\!=} \begin{bmatrix}e&e-1\\0&1\end{bmatrix}.$$

So: *No two of* e^Ae^B, e^Be^A, e^{A+B} *are equal.* ♦

33: Lemma. *Consider a mystery vector-valued function*

$$Z(t) = \begin{bmatrix}x_1(t)\\x_2(t)\\\vdots\\x_N(t)\end{bmatrix}.$$

Suppose Z *satisfies* $Z' = R \cdot Z$, *where* R *is an* $N\times N$ *matrix of numbers. Then each column,* Y, *of* e^{tR} *satisfies* $Y' = R{\cdot}Y$. *Hence the soln to* $Z' = RZ$ *is*

33a: $$Z(t) = e^{tR} \cdot Z(0).$$ ◊

34.1: *Diagonalizable Example.* Unknown fncs $x{=}x(t)$ and $y{=}y(t)$ satisfy

34.2: $$x' = -5x + 9y \quad\text{and}$$
$$y' = -6x + 10y.$$

So the coeff-matrix is $R := \begin{bmatrix}-5&9\\-6&10\end{bmatrix}$. Magic [or a nice guy] produces a conjugating matrix $U := \begin{bmatrix}3&1\\2&1\end{bmatrix}$ s.t

$$D := U^{-1}RU \overset{\text{note}}{=\!=} \begin{bmatrix}1&0\\0&4\end{bmatrix}$$

is a diagonal matrix.[♡15] Hence $e^{tR} = U e^{tD} U^{-1}$. I.e,

34.3: $$e^{tR} = U\begin{bmatrix}e^t&0\\0&e^{4t}\end{bmatrix}U^{-1}$$
$$\overset{\text{note}}{=\!=} \begin{bmatrix}3e^t-2e^{4t} & -3e^t+3e^{4t}\\ 2e^t-2e^{4t} & -2e^t+3e^{4t}\end{bmatrix}.$$

Our general soln, parameterized by numbers α and β, is

‡: $$x_{\alpha,\beta}(t) = [3e^t - 2e^{4t}]{\cdot}\alpha + [-3e^t + 3e^{4t}]{\cdot}\beta,$$
$$y_{\alpha,\beta}(t) = [2e^t - 2e^{4t}]{\cdot}\alpha + [-2e^t + 3e^{4t}]{\cdot}\beta.$$

As they must, $\alpha = x(0)$ and $\beta = y(0)$. □

35.1: *Nilpotent Example.* UFs $x = x(t)$ and $y = y(t)$ satisfy

35.2: $$x' = 2x - y \quad\text{and}$$
$$y' = 4x - 2y.$$

Hence the coeff-matrix is $R := \begin{bmatrix}2&-1\\4&-2\end{bmatrix}$. Note $R^2 = \mathbf{0}$. [I.e, R has nilpotency-degree 2.] Thus

35.3: $$e^{tR} = \mathbf{I} + tR \overset{\text{note}}{=\!=} \begin{bmatrix}1+2t & -t\\ 4t & 1-2t\end{bmatrix}.$$

Therefore, the soln to (35.2) is

‡: $$\begin{bmatrix}x(t)\\y(t)\end{bmatrix} = \begin{bmatrix}1+2t & -t\\ 4t & 1-2t\end{bmatrix}\cdot\begin{bmatrix}x(0)\\y(0)\end{bmatrix}.$$ □

[♡15] Note that $U^{-1} = \begin{bmatrix}1&-1\\-2&3\end{bmatrix}$

Defn. The ***characteristic polynomial*** of an $N\times N$ matrix M is

36.1: $$\wp_{\mathsf{M}}(z) \;:=\; \operatorname{Det}(\mathsf{M} - z\mathbf{I})$$

And the ***trace*** of M is

36.2: $$\operatorname{Trace}(\mathsf{M}) \;:=\; \begin{bmatrix}\text{Sum of elements on}\\ \text{main diagonal of } \mathsf{M}\end{bmatrix}.$$

Consider $\mathsf{Q} := \begin{bmatrix}\mathbf{a} & \mathbf{b}\\ \mathbf{c} & \mathbf{d}\end{bmatrix}$. Then $\operatorname{Trace}(\mathsf{Q}) = [\mathbf{a}+\mathbf{d}]$. And $\mathsf{Q} - z\mathbf{I} = \left[\begin{array}{c|c}\mathbf{a}-z & \mathbf{b}\\ \mathbf{c} & \mathbf{d}-z\end{array}\right]$. Hence

36.3: $$\wp_{\mathsf{Q}}(z) \;=\; z^2 - [\mathbf{a}+\mathbf{d}]z + [\mathbf{ad}-\mathbf{bc}] \;\stackrel{\text{note}}{=\!=}\; z^2 - \operatorname{Trace}(\mathsf{Q})\cdot z + \operatorname{Det}(\mathsf{Q}).$$

For a general $N\times N$ matrix M: If we write

$$\wp_{\mathsf{M}}(z) \;=\; [-1]^N z^N + \mathbf{\Omega}_{N-1}z^{N-1} + \ldots + \mathbf{\Omega}_0,$$

then $\mathbf{\Omega}_0 = \operatorname{Det}(\mathsf{M})$ and $\mathbf{\Omega}_{N-1} = [-1]^{N-1}\cdot\operatorname{Trace}(\mathsf{M})$. I.e,

36.4:
$$\wp_{\mathsf{M}}(z) \;=\; [-1]^N z^N + [-1]^{N-1}\operatorname{Trace}(\mathsf{M})z^{N-1} + \mathbf{\Omega}_{N-2}z^{N-2} + \ldots + \mathbf{\Omega}_1 z + \operatorname{Det}(\mathsf{M}).$$

Over $\mathbb{C}$, our char-poly factors as

$$\wp_{\mathsf{M}}(z) \;=\; [-1]^N\cdot[z-\alpha_1]\cdot[z-\alpha_2]\cdots[z-\alpha_N].$$

This list $\alpha_1,\alpha_2,\ldots,\alpha_N$ of (possibly complex) numbers is the list of ***eigenvalues*** of M. If M is diagonalizable, then

$$\mathsf{M} \sim \begin{bmatrix}\alpha_1 & & \\ & \ddots & \\ & & \alpha_N\end{bmatrix}.$$

Moreover, the *only* diagonal matrices to which M is similar are those whose main diagonal is some permutation of $\alpha_1,\ldots,\alpha_N$. □

36.5: Distinct-roots Thm. *Suppose that the char-poly*

$$\wp_{\mathsf{R}}(z) \;=\; [z-\beta_1]\cdot[z-\beta_2]\cdots[z-\beta_N]\cdot[-1]^N$$

of $N\times N$ matrix R has N distinct (possibly complex) roots. $\beta_1,\ldots,\beta_N$.[♡16] *Then R is indeed similar*[♡17] *to diagonal matrix* $\begin{bmatrix}\beta_1 & & \\ & \ddots & \\ & & \beta_N\end{bmatrix}$.

In particular, for column-vector $\mathsf{Z}(t) := \begin{bmatrix}x_1(t)\\ \vdots\\ x_N(t)\end{bmatrix}$ satisfying DE $\mathsf{Z}' = \mathsf{RZ}$, each $x_j(t)$ is simply a linear-combination of exponentials $\mathrm{e}^{\beta_1 t},\ldots,\mathrm{e}^{\beta_N t}$.

Letting $\mathbf{m}$ denote the maximum of the real-parts of the eigenvalues, it follows that no $x_j(t)$ can grow faster than [constant times $\mathrm{e}^{\mathbf{m}\cdot t}$], as $t\nearrow\infty$. ♢

[♡16] Recall, these are the *eigenvalues* of matrix R.
[♡17] Alas, it may be difficult to compute a conjugating matrix.

36.6: *Example.* Consider $\mathsf{X}'(t) = \mathsf{B}\cdot\mathsf{X}(t)$, where,

$$\mathsf{B} \;:=\; \begin{bmatrix}115 & 207 & -54\\ -72 & -130 & 34\\ -24 & -45 & 13\end{bmatrix}.$$

The char-poly of B is

$$\wp_{\mathsf{B}}(z) \;=\; -[z+5]\cdot[z^2-3z+8].$$

The discriminant of quadratic $q(z) := z^2-3z+8$ is $\operatorname{Discr}(g) = [-3]^2 - 4\cdot1\cdot8 = -23$. The roots of q are thus

$$S \;:=\; [3+\sqrt{23}\,\mathbf{i}]/2 \quad\textit{and}\quad \overline{S} \;\stackrel{\text{note}}{=\!=}\; [3-\sqrt{23}\,\mathbf{i}]/2.$$

So $\wp_{\mathsf{B}}(z) = -[z - -5][z-S][z-\overline{S}]$ in std form.

Since the three $\wp_{\mathsf{B}}$-roots are distinct, the Distinct-roots thm tell us that B is similar to diagonal matrix

$$\begin{bmatrix}-5 & & \\ & S & \\ & & \overline{S}\end{bmatrix}.$$

So each entry in $\mathsf{X}(t)$ is a lin-comb of $\mathrm{e}^{-5t},\mathrm{e}^{St},\mathrm{e}^{\overline{S}t}$. The max of the real-parts of $-5, S, \overline{S}$ is $\frac{3}{2}$. As $t\nearrow\infty$, then, no soln grows faster than $\operatorname{Const}\cdot\exp(\frac{3}{2}t)$. □

Recoding: Exchanging dimension for DE-order

For numbers $\boldsymbol{\Omega}_k \in \mathbb{C}$ and U.F $x{=}x(t)$,

37a: $$x^{(N)} = \sum_{k=0}^{N-1} [\boldsymbol{\Omega}_k \cdot x^{(k)}].$$

is an N^{th}-order DE in 1-dim'al space. Define col-vec

$$\mathsf{Z}(t) := \begin{bmatrix} x(t) \\ x'(t) \\ \vdots \\ x^{(N-2)}(t) \\ x^{(N-1)}(t) \end{bmatrix},$$

which is $N{\times}1$. We can restate (37a) as

37b: $\mathsf{Z}' = \mathsf{R} \cdot \mathsf{Z}$, where R is $N{\times}N$ matrix[♡18]

37c: $$\mathsf{R} := \begin{bmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & & \ddots & \ddots & \\ & & & & 0 & 1 \\ \boldsymbol{\Omega}_0 & \boldsymbol{\Omega}_1 & \boldsymbol{\Omega}_2 & \dots & \boldsymbol{\Omega}_{N-2} & \boldsymbol{\Omega}_{N-1} \end{bmatrix}.$$

[The unshown entries are zero. The cyan entries form the main diagonal.] The solution to (37a,37b) is

$$\mathsf{Z}(t) = \mathsf{e}^{t\cdot\mathsf{R}} \cdot \mathsf{Z}(0) = \exp(t{\cdot}\mathsf{R}) \cdot \mathsf{Z}(0).$$

But of course, we can solve (37a) with CCLDE, and do not need the matrix-exp. Here is a more interesting example:

37d: *Recoding Example.* Imagine U.Fs $x{=}x(t)$ and $y{=}y(t)$ related by DEs

37a†: $$x''' - 2x'' - 3x + 4y = 0, \quad \textit{and}$$
$$y' + 5x'' + 6x' + 7x - 8y = 0.$$

We can cheerfully recode this system as a 1st-order DE in 3+1 = 4 dim'al space, with U.F $\mathsf{Z}{=}\mathsf{Z}(t)$, as follows.

37b†: *Note* $\mathsf{Z}' = \mathsf{R}{\cdot}\mathsf{Z}$, *where* $\mathsf{Z} := \begin{bmatrix} y \\ x \\ x' \\ x'' \end{bmatrix}$ *and*

37c†: $$\mathsf{R} := \begin{bmatrix} 8 & -7 & -6 & -5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 3 & -0 & 2 \end{bmatrix}.$$

Hence the soln to (37a†, 37b†) is $\mathsf{Z}(t) = \mathsf{e}^{t\cdot\mathsf{R}} \cdot \mathsf{Z}(0)$.

In this instance, $\mathsf{e}^{t\cdot\mathsf{R}}$ is not so easy to compute, but it can be polynomially approximated by, say,

$$\exp(t{\cdot}\mathsf{R}) \approx \sum_{k=0}^{50} \left[t^k \mathsf{R}^k \middle/ k! \right],$$

with easily computable error-bounds. □

Aside. Into WolframAlpha, typing

```
{{8,-7,-6,-5},{0,0,1,0},{0,0,0,1},{-4,3,-0,2}}
```

```
i.e     {{ 8, -7, -6, -5},
         { 0,  0,  1,  0},
         { 0,  0,  0,  1},
         {-4,  3, -0,  2}}
```

indicates that $\wp_{\mathsf{R}}()$ has two real eigenvalues and a complex-conjugate pair of eigenvalues. As $t \nearrow \infty$, the growth rate of every soln is absolute-bnded by $\text{Const}+\text{Const}{\cdot}\exp(10.7 \cdot t)$.□

[♡18] See "Companion matrix" in Wikipedia.

MacFOLDE

Let's generalize.

38: Product-rule Lemma. *Suppose $\mathsf{A}(t)$ is a $J{\times}K$ matrix, and $\mathsf{B}(t)$ is a $K{\times}N$ matrix, each differentiable fncs. Then $J{\times}N$ matrix $\mathsf{P}(t) := \mathsf{A}(t)\cdot\mathsf{B}(t)$ is differentiable, and*

$$\mathsf{P}'(t) = [\mathsf{A}'(t){\cdot}\mathsf{B}(t)] + [\mathsf{A}(t){\cdot}\mathsf{B}'(t)]. \qquad \diamond$$

N.B. I.e, $\mathsf{P} = [\mathsf{A}'\,\mathsf{B}] + [\mathsf{A}\,\mathsf{B}']$. Matrix-mult is not commutative, so it is *likely* that P *fails* to equal, e.g, $[\mathsf{B}\mathsf{A}'] + [\mathsf{A}\mathsf{B}']$. □

39.1: *Warning!* Consider the matrix-valued fnc,

$$\mathsf{B}(t) := \begin{bmatrix} 3 & 2t \\ 0 & 0 \end{bmatrix}, \quad \text{so} \quad \mathsf{B}'(t) = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}.$$

Observe that

$$\mathsf{B}(t){\cdot}\mathsf{B}'(t) = \begin{bmatrix} 0 & 6 \\ 0 & 0 \end{bmatrix}, \quad \text{yet} \quad \mathsf{B}'(t){\cdot}\mathsf{B}(t) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Consequently, $\mathsf{B}'(t)$ does *not* commute with $\mathsf{B}(t)$. In symbols, $\mathsf{B}' \not\leftrightarrows \mathsf{B}$. □

39.2: Lemma. *Consider a differentiable matrix-valued function $\mathsf{B}(t)$ where, for each t, our $\mathsf{B}(t)$ is an $N{\times}N$ matrix. At each time t, suppose $\mathsf{B}'(t) \leftrightarrows \mathsf{B}(t)$. Then*

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\,\mathsf{e}^{\mathsf{B}(t)} = \mathsf{B}'(t)\cdot\mathsf{e}^{\mathsf{B}(t)} = \mathsf{e}^{\mathsf{B}(t)}\cdot\mathsf{B}'(t). \qquad \diamond$$

With C a matrix of numbers, and $\mathsf{B}(t) := \mathsf{C}\cdot t$, note that $\mathsf{B}'(t) = \mathsf{C}$. Hence $\mathsf{B}'(t)$ *does* commute with $\mathsf{B}(t)$.

This "constant coefficient" case is the case that interests us, so I call the following the ***Matrix-CC-FOLDE*** algorithm, abbreviated ***MacFOLDE***, even though the algorithm *does* apply whenever, for each t, matrix $\mathsf{B}'(t)$ commutes with $\mathsf{B}(t)$.

Step MFOL 0. We have U.F $\mathsf{Z}{=}\mathsf{Z}(t)$ which is a time-varying $N{\times}1$ matrix. Write the DE in the form

$$\text{40a:} \qquad \frac{\mathrm{d}\mathsf{Z}}{\mathrm{d}t} + [\mathsf{C}\cdot\mathsf{Z}] = \mathsf{G}(t),$$

where C is an $N{\times}N$ matrix of numbers, and $\mathsf{G}(t)$ is an $N{\times}1$ time-varying fnc. An antiderivative of C is $\mathsf{B}(t) := t{\cdot}\,\mathsf{C}$.

Define *multiplier function*

$$\text{40b:} \qquad \mathsf{M}(t) := \mathsf{e}^{\mathsf{B}(t)} \stackrel{\text{note}}{=\!=} \mathsf{e}^{t\mathsf{C}}.$$

Observe that $\mathsf{M}'(t) = \mathsf{M}(t)\cdot\mathsf{C}$. By (38), then,

$$\begin{aligned} [\mathsf{M}(t)\cdot\mathsf{Z}]' &= [\mathsf{M}(t){\cdot}\,\mathsf{C}\cdot\mathsf{Z}] + [\mathsf{M}(t)\cdot\mathsf{Z}'] \\ {**}: \quad &= \mathsf{M}(t)\cdot\Big[[\mathsf{C}{\cdot}\mathsf{Z}] + \mathsf{Z}'\Big] \\ &= \mathsf{M}(t)\cdot\mathsf{G}(t). \end{aligned}$$

Step MFOL 1. Define the column-vector *function* $\mathsf{P}(t) := \mathsf{M}(t){\cdot}\mathsf{G}(t)$, then compute

$$\mathsf{Q}(t) := \int^t \mathsf{P}().$$

For an arbitrary column-vec $\vec{\boldsymbol{\alpha}} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{bmatrix}$ of numbers, [*where* $\mathsf{M}{=}\mathsf{M}(t)$, $\mathsf{Z}{=}\mathsf{Z}(t)$, $\mathsf{Q}{=}\mathsf{Q}(t)$]

$$\mathsf{M}\cdot\mathsf{Z} = \vec{\boldsymbol{\alpha}} + \mathsf{Q}.$$

Multiplying by $\mathsf{M}^{-1} \stackrel{\text{note}}{=\!=} \mathsf{e}^{-t\mathsf{C}}$, and putting the t back in the notation, we have that

$$\text{40c:} \quad \underbrace{\mathsf{Z}_{\vec{\boldsymbol{\alpha}}}(t)}_{N\times 1} = \underbrace{\mathsf{e}^{-t\mathsf{C}}}_{N\times N}\cdot\Big[\underbrace{\vec{\boldsymbol{\alpha}}}_{N\times 1} + \underbrace{\mathsf{Q}(t)}_{N\times 1}\Big].$$

And if we arrange that $\mathsf{Q}(0) = \vec{\mathbf{0}}$, by defining

$$\mathsf{Q}(t) := \int_0^t \mathsf{P}(), \qquad \textit{then}$$

$$\text{40d:} \quad \underbrace{\mathsf{Z}(t)}_{N\times 1} = \underbrace{\mathsf{e}^{-t\mathsf{C}}}_{N\times N}\cdot\Big[\underbrace{\mathsf{Z}(0)}_{N\times 1} + \underbrace{\mathsf{Q}(t)}_{N\times 1}\Big].$$

Aside: Since C is constant, our $\mathsf{e}^{-t\mathsf{C}}$ is simply $\mathsf{M}(-t)$.

41.1: *Revisiting* (35.1), *from* P.55. Imagine unknown fncs $x = x(t)$ and $y = y(t)$ satisfying system

41.2:
$$\begin{aligned} x' &= 2x - y && \text{and} \\ y' &= 4x - 2y + 2. \end{aligned}$$

Setting $\mathsf{Z}(t) := \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ and $\mathsf{C} := \begin{bmatrix} -2 & 1 \\ -4 & 2 \end{bmatrix}$ and $\mathsf{G}(t) := \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, we can rewrite (41.2) as

$*$: $$\mathsf{Z}' + \mathsf{C}{\cdot}\mathsf{Z} = \mathsf{G}.$$

With this Z and C, our (35.2) example from page 55, was $\mathsf{Z}' + \mathsf{C}{\cdot}\mathsf{Z} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. As before, $\text{NilDeg}(\mathsf{C}) = 2$. Thus

40b†: $$\mathsf{M}(t) := \mathsf{e}^{t\mathsf{C}} = \mathbf{I} + t\mathsf{C} = \left[\begin{array}{c|c} 1 - 2t & t \\ -4t & 1 + 2t \end{array}\right],$$

since C is negative the R from (35.1). Computing,

$$\mathsf{P} := \mathsf{M}{\cdot}\mathsf{G} = \left[\begin{array}{c|c} 1 - 2t & t \\ -4t & 1 + 2t \end{array}\right]{\cdot}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2t \\ 2 + 4t \end{bmatrix}.$$

Integrating

$$\mathsf{Q} := \int_0^t \mathsf{P} = \begin{bmatrix} t^2 \\ 2t + 2t^2 \end{bmatrix} \overset{\text{note}}{=\!=} t{\cdot}\begin{bmatrix} t \\ 2 + 2t \end{bmatrix}.$$

In preparation for (40d), product $\mathsf{e}^{-t\mathsf{C}}{\cdot}\frac{1}{t}\mathsf{Q}$ equals

$$\overbrace{\left[\begin{array}{c|c} 1 + 2t & -t \\ 4t & 1 - 2t \end{array}\right]}^{\mathsf{e}^{-t\mathsf{C}} = \mathsf{M}(-t)} \cdot \begin{bmatrix} t \\ 2 + 2t \end{bmatrix} \overset{\text{note}}{=\!=} \begin{bmatrix} -t \\ 2 - 2t \end{bmatrix}.$$

Thus
$$\mathsf{e}^{-t\mathsf{C}} \cdot \mathsf{Q} = \begin{bmatrix} -t^2 \\ 2t - 2t^2 \end{bmatrix}.$$

With initial condition $x(0) = 0 = y(0)$, then,

$$\begin{aligned} x(t) &= -t^2, && \textit{and} \\ y(t) &= 2t - 2t^2. \end{aligned}$$

So the gen.soln to (41.2) is $\mathsf{e}^{-t\mathsf{C}}\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} + \mathsf{e}^{-t\mathsf{C}}\mathsf{Q}$, i.e

‡: $$\begin{aligned} x(t) &= [1 + 2t]{\cdot}x(0) - t{\cdot}y(0) - t^2, && \textit{and} \\ y(t) &= 4t{\cdot}x(0) + [1 - 2t]{\cdot}y(0) + [2t - 2t^2]. \end{aligned}$$

Compare this with (35.1 ‡), on P.55. □

§A Appendix: Misc examples

These may be cited from anywhere.

42: *Poly-coeffs yet ∃soln not* $\mathbf{C}^2$. Find a non-$\mathbf{C}^2$ function $y = y(t)$ that, for $t \in \mathbb{R}$, satisfies

42a: $$y'y + y^2 = G, \quad \text{where} \quad G(t) := t^4 - 2t^3 + 2t - 1.$$

ASIDE: This DE has form $P{\cdot}y'y + Q{\cdot}y^2 = G$. The coeff-fncs P,Q and target-fnc G are $\mathbf{C}^\infty$; indeed, *polynomials*; and P,Q are *constant*. Nonetheless, this DE admits a soln that is not even twice-differentiable. □

Soln. EASY SCAN: The DiffOp is invariant under negation; if f is a soln, then so is $-f$.

Could a degree-N poly satisfy (42a)? Well, the y^2 term forces $N \geq 2$. Thus $\text{Deg}(y'{\cdot}y) = 2N - 1$ and $\text{Deg}(y^2) = 2N$, so N must *be* 2. The method of UNDETERMINED COEFFS applies and we find that

42b: $$f(t) := [t-1]^2$$

satisfies (42a). Thus $-[t-1]^2$ is also a soln.

IDEA: The 0^{th} and 1^{st} derivatives of these solns *agree* at t=1, which are the only derivatives used by the DiffOp. So: *At* t=1, *we can stitch these solns together*. This gives this *new* soln:

†: $$y(t) := \begin{Bmatrix} +[t-1]^2 & \text{if } t \geq 1 \\ -[t-1]^2 & \text{if } t < 1 \end{Bmatrix} \stackrel{\text{note}}{=\!=} \left|t-1\right| \cdot [t-1].$$

Its derivative,

$$y'(t) = 2 \cdot \left|t-1\right|,$$

fails to be differentiable at t=1. So (†) is not twice-differentiable, hence not $\mathbf{C}^2$.

Let's check that (†) satisfies (42a). Computing,

$$y' \cdot y = 2{\cdot}[t-1]^3 = 2t^3 - 6t^2 + 6t - 2,$$
$$y^2 = [t-1]^4 = t^4 - 4t^3 + 6t^2 - 4t + 1.$$

Adding these together produces (42a). ♦

42c: *N.B:.* Our three fncs, (†) and $\pm[t-1]^2$, each solve first-order DE (42a), *and*: Their 0^{th} *and* 1^{st} derivatives agree at t=1. So even possession of *two* initial conditions to a first-order DE, need not be sufficient to uniquely specify a soln.

ASIDE: Our $G(t)$ factors as $[t-1]^3 \cdot [t+1]$. □

A FLEA AND A FLY IN A FLUE

Were imprisoned, so what could they do?
Said the fly, "let us flee!"
Said the flea, "let us fly!"
So they flew through a flaw in the flue.

–Ogden Nash

§B Binomial coeffs & the Product rule

For a natnum n, use "$n!$" to mean **"n *factorial*"**; the product of all posints $\leq n$. So $3! = 3 \cdot 2 \cdot 1 = 6$ and $5! = 120$. Also $0! = 1 = 1!$.

For natnum B and arb. complex number α, define

Rising Fctrl: $[\![\alpha \uparrow \mathsf{B}]\!] := \alpha \cdot [\alpha+1] \cdot [\alpha+2] \cdots [\alpha + [\mathsf{B}-1]]$,
Falling Fctrl: $[\![\alpha \downarrow \mathsf{B}]\!] := \alpha \cdot [\alpha-1] \cdot [\alpha-2] \cdots [\alpha - [\mathsf{B}-1]]$.

E.g, $[\![\mathsf{B}\downarrow\mathsf{B}]\!] = \mathsf{B}! = [\![1\uparrow\mathsf{B}]\!]$. Two further examples,

$$\left[\!\!\left[\tfrac{2}{7} \downarrow 4\right]\!\!\right] = \frac{2}{7} \cdot \frac{-5}{7} \cdot \frac{-12}{7} \cdot \frac{-19}{7} \quad \textit{and} \quad [\![1 \downarrow 3]\!] = 1 \cdot 0 \cdot {-1} = 0\,.$$

In particular, for $n \in \mathbb{N}$: If $\mathsf{B} > n$ then $[\![n \downarrow \mathsf{B}]\!] = 0$. We pronounce $[\![5 \downarrow \mathsf{B}]\!]$ as "5 falling-factorial B".

Binomial. The ***binomial coefficient*** $\binom{7}{3}$, read "7 choose 3", means *the number of ways of choosing* 3 *objects from* 7 *distinguishable objects*. Emphasising putting 3 objects in our left pocket and the remaining 4 in our right, we may write the coeff as $\binom{7}{3,4}$. [Read as "7 choose 3-comma-4."] Evidently

$$\dagger: \quad \binom{\mathsf{N}}{j} \overset{\text{with } k := \mathsf{N}-j}{=\!=\!=\!=} \binom{\mathsf{N}}{j,\ k} = \frac{\mathsf{N}!}{j! \cdot k!} = \frac{[\![\mathsf{N} \downarrow j]\!]}{j!}.$$

Note $\binom{7}{0} = \binom{7}{0,7} = 1$. Finally, the Binomial theorem says

$$\pounds: \quad [x+y]^{\mathsf{N}} = \sum_{j+k=\mathsf{N}} \binom{\mathsf{N}}{j,k} \cdot x^j y^k\,,$$

where (j,k) ranges over all *ordered* pairs of natural numbers with sum N.

For natnum N, binomials satisfy this addition law:

$$*: \quad \binom{\mathsf{N}+1}{\mathsf{B}+1} = \overbrace{\binom{\mathsf{N}}{\mathsf{B}}}^{\text{Pick last object.}} + \overbrace{\binom{\mathsf{N}}{\mathsf{B}+1}}^{\text{Avoid last object.}}\,.$$

Extending this to *all* $\mathsf{B} \in \mathbb{Z}$ forces:

$$\binom{\mathsf{N}}{\mathsf{B}} = 0, \quad \begin{array}{l} \textit{when } \mathsf{B} > \mathsf{N} \\ \textit{or } \mathsf{B} \textit{ negative.} \end{array}$$

Case $\mathsf{B} > \mathsf{N}$ is automatic in formula $\binom{\mathsf{N}}{\mathsf{B}} = \frac{[\![\mathsf{N}\downarrow\mathsf{B}]\!]}{\mathsf{B}!}$.

Multinomial. In general, for natural numbers $\mathsf{N} = k_1 + \ldots + k_P$, the ***multinomial coefficient*** $\binom{\mathsf{N}}{k_1,\, k_2,\, \ldots,\, k_P}$ is the number of ways of partitioning N objects, by putting k_1 objects in pocket-one, k_2 objects in pocket-two, …putting k_P objects in the P^{th} pocket. Easily

$$\ddagger: \quad \binom{\mathsf{N}}{k_1, k_2, \ldots, k_P} = \frac{\mathsf{N}!}{k_1! \cdot k_2! \cdot \ldots \cdot k_P!}\,.$$

Unsurprisingly, $[x_1 + \ldots + x_P]^{\mathsf{N}}$ equals the sum of terms

$$\pounds\pounds: \quad \binom{\mathsf{N}}{k_1,\ldots,k_P} \cdot {x_1}^{k_1} \cdot {x_2}^{k_2} \cdots {x_P}^{k_P}\,,$$

taken over all natnum-tuples $\vec{k} = (k_1, \ldots, k_P)$ that sum to N. [That is multinomial analog of the Binomial Thm.]

Define the sum $S_\ell := k_1 + k_2 + \ldots + k_\ell$. Then multinomial LhS(‡) equals this product of binomials:

$$\binom{\mathsf{N}}{k_1} \cdot \binom{\mathsf{N} - S_1}{k_2} \cdot \binom{\mathsf{N} - S_2}{k_3} \cdot \ldots \cdot \binom{\mathsf{N} - S_{P-1}}{k_P}.$$

[The last term is $\binom{k_P}{k_P} \overset{\text{note}}{=\!=} 1$.]

Calculus applications

Bi/Multi-nomials appear in differentiation formulas.

43a: Product Rule. *For natnum N, and N-times differentiable functions f and g:*

$$*: \quad [f \cdot g]^{(N)} = \sum_{j+k=N} \binom{N}{j,\ k} \cdot f^{(j)} \cdot g^{(k)}\,,$$

*where (j,k) ranges over all **ordered pairs** of natural numbers with sum N.* ♢

E.g: $[f \cdot g]^{(4)} = fg^{(4)} + 4f^{(1)}g^{(3)} + 6f^{(2)}g^{(2)} + 4f^{(3)}g^{(1)} + f^{(4)}g$.

43b: Lemma. *For posints N, J, K with $J+K = N+1$,*

$$¥: \quad \binom{N}{J-1,\ K} + \binom{N}{J,\ K-1} = \binom{N+1}{J,\ K}\,. \quad ♢$$

Proof. The LhS(¥) equals

$$\frac{J}{J} \cdot \frac{N!}{[J-1]!\ K!} + \frac{N!}{J!\ [K-1]!} \cdot \frac{K}{K} = \frac{[J+K] \cdot N!}{J!\ K!}\,,$$

which equals RhS(¥). ♦

Pf of (43a). At $N{=}0$, our $(*)$ says $fg = fg$; a tautology. Fixing N for which $(*)$ holds, note $[f\cdot g]^{(N+1)}$ equals $\sum_{j+k=N}\binom{N}{j,k}[f^{(j)}\cdot g^{(k)}]'$, which equals

$$\overbrace{\sum_{j+k=N}\binom{N}{j,k} f^{(j+1)}g^{(k)}}^{A} + \overbrace{\sum_{j+k=N}\binom{N}{j,k} f^{(j)}g^{(k+1)}}^{B}.$$

Letting $J := j{+}1$ and $K := k$, rewrite A as

$$\dagger\colon\qquad A = \sum_{\substack{J+K=N+1,\\ J\geq 1}} \binom{N}{J-1,\ K}\cdot f^{(J)}g^{(K)}.$$

Similarly, with $K := k{+}1$ and $J := j$, rewrite B as

$$\ddagger\colon\qquad B = \sum_{\substack{J+K=N+1,\\ K\geq 1}} \binom{N}{J,\ K-1}\cdot f^{(J)}g^{(K)}.$$

Separating out the $K{=}0$ term from $(\dagger)$ and the $J{=}0$ term from $(\ddagger)$, says that $A+B$ equals

$$\binom{N}{N,0} f^{(N+1)}g^{(0)} + \binom{N}{0,N} f^{(0)}g^{(N+1)} + \sum_{\substack{J+K=N+1,\\ J,K\geq 1}} \left[\binom{N}{J-1,\ K} + \binom{N}{J,\ K-1}\right]\cdot f^{(J)}g^{(K)}.$$

Use the lemma, (¥), to rewrite the summand. Thus $A+B$ equals

$$f^{(N+1)}g^{(0)} + f^{(0)}g^{(N+1)} + \sum_{\substack{J+K=N+1,\\ J,K\geq 1}} \binom{N+1}{J,\ K}\cdot f^{(J)}g^{(K)}.$$

And this equals $\sum_{j+k=N+1}\binom{N+1}{j,\,k}\cdot f^{(j)}g^{(k)}$, as desired. ♦

Larger product. Given a tuple $\mathbf{J} = (j_1,\dots,j_P)$ of natnums, let $+\mathbf{J} := j_1+\dots+j_P$. With $N := +\mathbf{J}$, let $\binom{N}{\mathbf{J}}$ mean multinomial coeff $\binom{N}{j_1,j_2,\dots,j_P}$. Finally, given a tuple $\vec{\mathbf{f}} := (f_1,\dots,f_P)$ of differentiable fncs, let $\vec{\mathbf{f}}^{(\mathbf{J})}$ abbreviate this product of derivatives:

$$\vec{\mathbf{f}}^{(\mathbf{J})} := f_1^{(j_1)}\cdot f_2^{(j_2)}\cdot\ldots\cdot f_P^{(j_P)}.$$

[When tuple $\mathbf{J}$ is used this way, it is called a ***multi-index***.]

43c: Gen. Product Rule. *Fix natnum N, posint P, and N-times differentiable functions, $\vec{\mathbf{f}} := (f_1,\dots,f_P)$. Then*

$$V_P\colon\qquad [f_1\cdot\ldots\cdot f_P]^{(N)} = \sum_{\mathbf{J}\,:\,+\mathbf{J}=N}\binom{N}{\mathbf{J}}\cdot\vec{\mathbf{f}}^{(\mathbf{J})}.\qquad\diamond$$

Proof. Eqn (V_1) asserts tautology $f_1^{(N)} = f_1^{(N)}$. We proceed by induction on P. Fixing P such that (V_P), we now establish (V_{P+1}).

Fix $P{+}1$ fncs $f_1,\dots,f_P,g$, and let $\Phi := f_1\cdot\ldots\cdot f_P$. Then $[f_1\cdot\ldots\cdot f_P\cdot g]^{(N)}$ is $[\Phi\cdot g]^{(N)}$. By $(*)$, it equals

$$*1\colon\qquad \sum_{s+k=N}\binom{N}{s,\ k}\cdot\Phi^{(s)}\cdot g^{(k)},$$

where (s,k) ranges over all natnum-pairs with sum N. Courtesy (V_P), our $\Phi^{(s)}$ equals

$$\sum_{\mathbf{J}\,:\,+\mathbf{J}=s}\binom{s}{\mathbf{J}}\cdot\vec{\mathbf{f}}^{(\mathbf{J})},\qquad \textit{where } \mathbf{J} = (j_1,\dots,j_P).$$

Plugging this in to $(*1)$ gives

$$*2\colon\qquad \sum_{s+k=N}\left[\sum_{\mathbf{J}:\,+\mathbf{J}=s}\binom{N}{s,\ k}\binom{s}{\mathbf{J}}\cdot\vec{\mathbf{f}}^{(\mathbf{J})}\cdot g^{(k)}\right].$$

But product $\binom{N}{s,\ k}\binom{s}{\mathbf{J}}$ equals multinomial $\binom{N}{j_1,\dots,j_P,k}$. Renaming k to j_{P+1}, and g to f_{P+1}, writes $(*2)$ as

$$\sum_{\substack{j_1+\ldots+j_P+j_{P+1}\\ =N}}\binom{N}{j_1,\ \ldots,\ j_{P+1}}\cdot f_1^{(j_1)}\cdot\ldots\cdot f_P^{(j_P)}\cdot f_{P+1}^{(j_{P+1})},$$

which indeed is RhS of (V_{P+1}). ♦

Deriv(product). Consider $f(t) := 3^t$, $g(t) := \sin(5t)$ and $h(t) := \mathrm{e}^{7t}$. The 6^{th}-derivative, $[f\cdot g\cdot h]^{(6)}$, is a sum of terms. *What is the coeff of the $f''\cdot g'\cdot h'''$ term?*

Soln. By the generalized product rule, (43c), the coefficient of $f^{(2)}g^{(1)}h^{(3)}$ is

$$\binom{6}{2,\ 1,\ 3} \overset{\text{note}}{=\!=} \binom{6}{2}\binom{4}{1}\binom{3}{3} = \frac{6\cdot 5}{2\cdot 1}\cdot\frac{4}{1}\cdot 1 = 60.$$

Continuing, note:

$$f^{(2)} = [\log(3)]^2\cdot f;\quad g^{(1)}(t) = 5\cos(5t);\quad h^{(3)} = 7^3\cdot h.$$

So one summand in the sum forming $[f\cdot g\cdot h]^{(6)}$, is

$$60\cdot\log(3)^2\cdot 5\cdot 7^3\cdot\left[3^t\cdot\cos(5t)\cdot\mathrm{e}^{7t}\right].\qquad ♦$$

§C Order-3 VoP

CC-VoP Example 2. U.F $s = s(t)$ satisfies

44a: $$s''' - s'' = t\mathrm{e}^t .$$

A good approach is to define $q := s''$, solve DE

¥: $$q' - q = G$$

where $G(t) := t\mathrm{e}^t$, then anti-diff twice. First-order DE (¥) can be solved via FOLDE (17a), or Poly-Exp (10a), or Convol-GenTar (25a) P.44, or VoP (26.4).

But for illustration, I'm going to solve the original (44a) by VoP. Set $E := \mathrm{e}^t$ and $\mathsf{L} := \mathbf{D}^3 - \mathbf{D}^2$.

VoP1. The aux.poly of L is $[z-0]^2[z-1]$. So $\{Y_0 := 1,\ Y_1 := t,\ Y_2 := \mathrm{e}^t\}$ is an L.I. triple of fncs annihilated by L.

VoP2. The Wronskian-matrix of $(1, t, \mathrm{e}^t)$ is

∗: $$\mathsf{M} := \begin{bmatrix} 1 & t & E \\ 0 & 1 & E \\ 0 & 0 & E \end{bmatrix}. \quad \textit{So}\ \ D := \mathrm{Det}(\mathsf{M}) = E .$$

We compute fncs h_0, h_1, h_2 satisfying matrix-eqn $\mathsf{M} \cdot \mathsf{H} = \mathsf{T}$, where

$$\mathsf{H} := \begin{bmatrix} h_0 \\ h_1 \\ h_2 \end{bmatrix} \quad \text{and} \quad \mathsf{T} := \begin{bmatrix} 0 \\ 0 \\ G \end{bmatrix}.$$

Cramer's thm has us examine matrices

$$\begin{bmatrix} 0 & t & E \\ 0 & 1 & E \\ G & 0 & E \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & E \\ 0 & 0 & E \\ 0 & G & E \end{bmatrix}, \quad \begin{bmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & G \end{bmatrix}.$$

Their determinants are, respectively,

$$GE[t-1], \qquad -GE, \qquad G .$$

Dividing each by the (∗)-Wronskian, E, gives

$$h_0 = G[t-1], \quad h_1 = -G, \quad h_2 = G/E .$$

VoP3. Computing anti-derivatives,

$$f_0 = \int h_0 = \int^t x\mathrm{e}^x \cdot [x-1]\,\mathrm{d}x = \mathrm{e}^t[t^2 - 3t + 3], \ \textit{and}$$

$$f_1 = \int h_1 = \int^t -x\mathrm{e}^x\,\mathrm{d}x = \mathrm{e}^t[1-t], \qquad \textit{and}$$

$$f_2 = \int h_2 = \int^t \frac{x\mathrm{e}^x}{\mathrm{e}^x}\,\mathrm{d}x = \frac{t^2}{2}.$$

So a soln to (44a) is

$$\begin{aligned} s &= f_0 Y_0 + f_1 Y_1 + f_2 Y_2 \\ &= \mathrm{e}^t[t^2 - 3t + 3] \cdot 1 + \mathrm{e}^t[1-t] \cdot t + \tfrac{t^2}{2} \cdot \mathrm{e}^t \\ &= \left[\tfrac{t^2}{2} - 2t + 3\right] \cdot \mathrm{e}^t . \end{aligned}$$

Recall $\mathsf{L}(\mathrm{e}^t) = 0$, so $\mathsf{L}(3\mathrm{e}^t) = 0$, and we may thus use the simpler soln

$$s = \left[\frac{t^2}{2} - 2t\right] \cdot \mathrm{e}^t$$

to (44a).

You have to do your own growing no matter how tall your grandfather was.
—*Abraham Lincoln*

§D It's about Brine, it's about Space, it's about brine moving place to place...

The rather cute theme song.

Remark. Brine is saline-water, NaCl in H_20.

The *Cascading tanks* on the next page is an instance of *Compartmental analysis.* □

Compartmental analysis [§3.2–NSS9]

Brine with $1.3\frac{\text{lb}}{\text{gal}}$ salt flows at rate $4\frac{\text{gal}}{\text{min}}$ into a tank that initially holds 12gal of $2\frac{\text{lb}}{\text{gal}}$-salt brine. The tank is well-mixed, and brine is flowing out at rate $4\frac{\text{gal}}{\text{min}}$. We seek a formula for $y(t)$, the number of lbs of salt in the tank at time t.

Henceforth, use italic boldface $\boldsymbol{0}$ to mean 0min.

Units:	Symbol:	Description:
lb	$y(t)$	Salt in tank @ t.
gal	$W(t)$	**W**ater in tank @ t.
	$\text{U} := W(\boldsymbol{0})$	Initial amo**u**nt of water.
lb/gal	S	Input **s**alinity.
	$\sigma(t)$	Salinity in tank @ t.
	$\text{D} := \sigma(\boldsymbol{0}) - \text{S}$	Initial **D**ifference in salinities.
gal/min	R	Input flow-**r**ate of water.
	$\boldsymbol{\rho}$	Output flow-**r**ate of water.
	$\text{A} := \text{R} - \boldsymbol{\rho}$	**A**ccumulation flow-rate.
	$\text{L} := \boldsymbol{\rho} - \text{R} = \text{-A}$	**L**oss flow-rate.
min	$\text{E} := \text{U}/\text{L}$	Time-to-**E**mpty, when $\text{L}>0\frac{\text{gal}}{\text{min}}$.
1/min	$\boldsymbol{\Gamma} := \frac{\text{R}}{\text{U}}$	A useful constant.

By definition of the quantities involved

45a: $$W(t) = \text{U} + \text{A}t \quad \text{and} \quad \sigma(t) = \frac{y(t)}{W(t)}.$$

Our salt-fnc y satisfies DE

45b: $$y'(t) = \overbrace{\text{R}\cdot\text{S}}^{\text{Input}} - \overbrace{\boldsymbol{\rho}\cdot\sigma(t)}^{\text{Output}} \stackrel{\text{note}}{=\!=} \text{SR} - \frac{\boldsymbol{\rho}}{W(t)}\cdot y(t).$$

To match our FOLDE notation, let

$$\text{G} := \text{SR} \quad \textit{and} \quad C(t) := \frac{\boldsymbol{\rho}}{W(t)}.$$

So we can re-write (45b) as

$$y'(t) + C(t)y(t) = \text{G}.$$

Case: $\text{R}=\boldsymbol{\rho}$, not zero.

Hence $C()$ is the constant $\boldsymbol{\Gamma} := \frac{\text{R}}{\text{U}} \neq 0$. Step (**F0**) of FOLDE has us anti-diff, then exponentiate, to get

45c: $$M(t) := \mathsf{e}^{\boldsymbol{\Gamma} t}.$$

Step (**F1**): Anti-diff'ing product $\text{G}\cdot\mathsf{e}^{\boldsymbol{\Gamma}t}$ gives

$$Q(t) := \frac{\text{G}}{\boldsymbol{\Gamma}}\cdot\mathsf{e}^{\boldsymbol{\Gamma} t} \stackrel{\text{note}}{=\!=} \text{SU}\cdot\mathsf{e}^{\boldsymbol{\Gamma} t}.$$

For an arb.constant α, then, step (**F2**) gives

45d: $$y(t) = \mathsf{e}^{\text{-}\boldsymbol{\Gamma} t}\cdot\left[\alpha + \text{SU}\cdot\mathsf{e}^{\boldsymbol{\Gamma} t}\right] = \frac{\alpha}{\mathsf{e}^{\boldsymbol{\Gamma} t}} + \text{SU}.$$

Divide through by U, and rename $\frac{\alpha}{\text{U}}$ to α [which is, after all, arbitrary] to get

$$\sigma(t) = \frac{\alpha}{\mathsf{e}^{\boldsymbol{\Gamma} t}} + \text{S}.$$

Solve for α, and re-order, to obtain that

45e: $$\sigma(t) = \text{S} + \frac{\text{D}}{\mathsf{e}^{[\frac{\text{R}}{\text{U}}\cdot t]}}.$$

Or use SoV. Alternatively, write (45b) as

$$\frac{\text{d}y}{\text{d}t} = \text{G} - \boldsymbol{\Gamma}\cdot y$$

and separate variables to get

$$\frac{1}{\text{G} - \boldsymbol{\Gamma}y}\cdot \text{d}y = 1\cdot \text{d}t.$$

Only considering when $\text{G} - \boldsymbol{\Gamma}y > 0$, we anti'diff to get

$$\frac{1}{\text{-}\boldsymbol{\Gamma}}\cdot\log(\text{G} - \boldsymbol{\Gamma}y) = t + \alpha,$$

using arb.constant α. Cross-mult then exponentiate to get $\text{G} - \boldsymbol{\Gamma}y = 1/\mathsf{e}^{\boldsymbol{\Gamma}t + \boldsymbol{\Gamma}\alpha}$. Replace $\mathsf{e}^{\text{-}\boldsymbol{\Gamma}\alpha}$ by $\text{-}\alpha$ [skipping some details] to get

$$\text{G} - \boldsymbol{\Gamma}y = \frac{\text{-}\alpha}{\mathsf{e}^{[\boldsymbol{\Gamma}\cdot t]}}.$$

Solve for $y{=}y(t)$, giving

$$y(t) = \frac{\alpha}{\mathsf{e}^{\boldsymbol{\Gamma}\cdot t}} + \frac{\text{G}}{\boldsymbol{\Gamma}} \stackrel{\text{note}}{=\!=} \frac{\alpha}{\mathsf{e}^{\boldsymbol{\Gamma} t}} + \text{SU}.$$

And this is RhS(45*d*).

Case: $\text{R}\neq\boldsymbol{\rho}$.

I.e, $\text{A} \neq 0$, so $W()$ is not constant.

∗: *In this section, we only consider values of t where $W(t) \stackrel{\text{note}}{=\!=} \text{U} + \text{A}t$ is positive.*

Step (**F0**): Anti-diff $C(t) = \frac{\boldsymbol{\rho}}{\text{U}+\text{A}t}$ to get

$$B(t) := \frac{\boldsymbol{\rho}}{\text{A}}\cdot\log(\text{U} + \text{A}t),$$

using (∗). Setting $\theta := \frac{\boldsymbol{\rho}}{\text{A}}$, then, exponentiating gives

$$M(t) = [\text{U} + \text{A}t]^{\theta}.$$

Step (**F1**): Anti-diff'ing product G$\cdot M(t)$ hands us

$$Q(t) \;:=\; \frac{\mathrm{G}}{\mathrm{A}\cdot[\theta+1]}\cdot[\mathrm{U} \,+\, \mathrm{A}t]^{\theta+1}\,.$$

Note $[\mathrm{A}\theta]+\mathrm{A}=\mathrm{R}$ and $\frac{\mathrm{G}}{\mathrm{R}}=\mathrm{S}$. Step (**F2**) has us add an arb.constant α, then divide by $M(t)$, giving

$$y(t) \;=\; \frac{1}{M(t)}\cdot\Big[\mathrm{S}\cdot[\mathrm{U} \,+\, \mathrm{A}t]^{\theta+1} \;+\; \alpha\Big]\,.$$

Dividing by $W(t) \stackrel{\text{note}}{=\!=} [\mathrm{U} \,+\, \mathrm{A}t]$ yields

$$\sigma(t) \;=\; \mathrm{S} \;+\; \frac{\alpha}{[\mathrm{U} \,+\, \mathrm{A}t]^{\mathrm{R}/\mathrm{A}}}\,,$$

since $\theta+1=\frac{\mathrm{R}}{\mathrm{A}}$. Dividing top and bottom by $[\mathrm{U}]^{\mathrm{R}/\mathrm{A}}$, and solve for α to arrive at this:

45f: $$\sigma(t) \;=\; \mathrm{S} \;+\; \frac{\mathrm{D}}{[1 \,+\, \frac{\mathrm{A}}{\mathrm{U}}\cdot t]^{\mathrm{R}/\mathrm{A}}}\,,$$

The A rate is positive:negative as the tank is filling:draining. When draining, it is convenient to express this formula in terms of the *Loss flow-rate*, L, and *time-to-Empty*, E. Since $\frac{\mathrm{A}}{\mathrm{U}}=\frac{-\mathrm{L}}{\mathrm{U}}=\frac{-1}{\mathrm{E}}$, our (45f) becomes

45g: $$\sigma(t) \;=\; \mathrm{S} \;+\; \mathrm{D}\cdot\Big[1 \;-\; \tfrac{1}{\mathrm{E}}\cdot t\Big]^{\mathrm{R}/\mathrm{L}}\,,$$

Plausibility. Soln (45f) handles when $\mathrm{A}\neq 0$. Do we get our A=0 soln, (45e), as a limit when we send A to zero? Let's check, by applying L'Hôpital's rule to the denominator of (45f). Let

$$\mathscr{L} \;:=\; \lim_{\mathrm{A}\to 0}\big[1 \,+\, \tfrac{\mathrm{A}}{\mathrm{U}}\cdot t\big]^{\mathrm{R}/\mathrm{A}}\,.$$

Since log is continuous, $\log(\mathscr{L})=\widehat{\mathscr{L}}$, where

$$\widehat{\mathscr{L}} \;:=\; \lim_{\mathrm{A}\to 0}\tfrac{\mathrm{R}}{\mathrm{A}}\cdot\log\big(1 \,+\, \tfrac{\mathrm{A}}{\mathrm{U}}\cdot t\big)\,.$$

Applying L'Hôpital's, L'Hôpital's rule

$$\lim_{\mathrm{A}\to 0}\frac{\log\big(1 \,+\, \frac{t}{\mathrm{U}}\cdot\mathrm{A}\big)}{\mathrm{A}} \stackrel{\text{l'Hôp}}{=\!=} \lim_{\mathrm{A}\to 0}\frac{\Big[\frac{1}{1+\frac{t}{\mathrm{U}}\cdot\mathrm{A}}\Big]\cdot\frac{t}{\mathrm{U}}}{1}$$
$$= \lim_{\mathrm{A}\to 0}\Big[\frac{t}{\mathrm{U} \,+\, t\cdot\mathrm{A}}\Big]$$
$$= \frac{t}{\mathrm{U} \,+\, [t\cdot 0]} \;=\; \frac{t}{\mathrm{U}}\,.$$

Hence $\widehat{\mathscr{L}}=\mathrm{R}\cdot\frac{t}{\mathrm{U}}$. Consequently

$$\mathscr{L} \;=\; \mathsf{e}^{\frac{\mathrm{R}}{\mathrm{U}}\cdot t}\,,$$

which indeed equals the denominator of (45e).

Cascading tanks

Calling the above tank "tank-1", we generalize to have tank-1 feed into tank-2, which feeds into tank-3 etc. Each tank has constant input and output flow-rate R. The amount of water in each tank is U.

Use $\sigma_N(t)$ for the salt-concentration in tank-N at time t, and use [recall that italic boldface $\boldsymbol{0}$ means 0min.]

$$\mathrm{D}_N \;:=\; \sigma_N(\boldsymbol{0}) \;-\; \mathrm{S}\,. \quad \text{As a convenience,}$$
$$\mathrm{D}_0 \;=\; \mathrm{S} \;-\; \mathrm{S} \stackrel{\text{note}}{=\!=} 0\tfrac{\mathsf{lb}}{\mathsf{gal}} \quad\text{and}$$
$$*: \quad \sigma_0(\cdot) \;\equiv\; \mathrm{S}\,,$$

by imagining that the source is an ∞-volume tank-0.

We will show, for $N=0,1,2,\ldots$, that[♡19]

45h:
$$\sigma_N(t) \stackrel{?}{=} \mathrm{S} \,+\, \frac{f_N(t)}{\mathsf{e}^{\boldsymbol{\Gamma}t}}\,, \quad\text{where}$$
$$f_N(t) \;:=\; \sum_{k=0}^{N}\tfrac{1}{k!}\cdot\mathrm{D}_{N-k}\cdot[\boldsymbol{\Gamma}t]^k$$
$$\stackrel{\text{note}}{=\!=} \sum_{k=0}^{N-1}\tfrac{1}{k!}\cdot\mathrm{D}_{N-k}\cdot[\boldsymbol{\Gamma}t]^k,$$

since D_0 is zero. To illustrate this defn:

$$\sigma_0(t) \;=\; \mathrm{S}\,;$$

$$\sigma_1(t) \;=\; \mathrm{S} \,+\, \frac{\mathrm{D}_1}{\mathsf{e}^{\boldsymbol{\Gamma}t}}\,;$$

$$\sigma_2(t) \;=\; \mathrm{S} \,+\, \frac{\mathrm{D}_1\boldsymbol{\Gamma}t \,+\, \mathrm{D}_2}{\mathsf{e}^{\boldsymbol{\Gamma}t}}\,;$$

$$\sigma_3(t) \;=\; \mathrm{S} \,+\, \frac{\frac{1}{2}\mathrm{D}_1[\boldsymbol{\Gamma}t]^2 \,+\, \mathrm{D}_2\boldsymbol{\Gamma}t \,+\, \mathrm{D}_3}{\mathsf{e}^{\boldsymbol{\Gamma}t}}\,;$$

$$\sigma_4(t) \;=\; \mathrm{S} \,+\, \frac{\frac{1}{6}\mathrm{D}_1[\boldsymbol{\Gamma}t]^3 \,+\, \frac{1}{2}\mathrm{D}_2[\boldsymbol{\Gamma}t]^2 \,+\, \mathrm{D}_3\boldsymbol{\Gamma}t \,+\, \mathrm{D}_4}{\mathsf{e}^{\boldsymbol{\Gamma}t}}\,.$$

N.B: The numerator in $\sigma_4(t)$ is

$$\frac{\mathrm{D}_1[\boldsymbol{\Gamma}t]^3}{3!} \,+\, \frac{\mathrm{D}_2[\boldsymbol{\Gamma}t]^2}{2!} \,+\, \frac{\mathrm{D}_3[\boldsymbol{\Gamma}t]^1}{1!} \,+\, \frac{\mathrm{D}_4[\boldsymbol{\Gamma}t]^0}{0!}\,.$$

[♡19]Note that $\mathrm{Deg}(f_N) \le N-1$, since D_0 is zero.

For future use, verify this recurrence relation:

$$**: \quad [f_{N+1}]' = \mathbf{\Gamma} \cdot f_N .$$

Specifically,

$$***: \quad f_{N+1}(t) = \mathrm{D}_{N+1} + \mathbf{\Gamma} \cdot \int_0^t f_N .$$

For convenience, we we restate...

45h:
$$\sigma_N(t) = \mathrm{S} + \frac{f_N(t)}{\mathsf{e}^{\mathbf{\Gamma} t}}, \quad \text{where}$$
$$f_N(t) := \sum_{k=0}^{N} \frac{1}{k!} \cdot \mathrm{D}_{N-k} \cdot [\mathbf{\Gamma} t]^k \stackrel{\text{note}}{=} \sum_{k=0}^{N-1} \frac{1}{k!} \cdot \mathrm{D}_{N-k} \cdot [\mathbf{\Gamma} t]^k,$$

Proving (45h)**.** Product $\mathbf{\Gamma} t$ is unitless, so $f_N(t)$ is in lb/gal; hence so is $\mathrm{S} + [f_N(t)/\mathsf{e}^{\mathbf{\Gamma} t}]$, as it should be.

Secondly $f_N(\boldsymbol{0}) = \frac{1}{0!} \cdot \mathrm{D}_{N-0} \cdot 1 \stackrel{\text{note}}{=} \mathrm{D}_N$. Thus $\mathrm{S} + \frac{f_N(0)}{\mathsf{e}^0}$ equals $\mathrm{S} + \mathrm{D}_N$, which indeed equals $\sigma_N(\boldsymbol{0})$, as it should. What remains, is for us to verify that (45h) satisfies the appropriate DE.

Base case. Note $f_0(\cdot) = \frac{1}{0!} \cdot \mathrm{D}_0 \stackrel{\text{note}}{=} 0 \frac{\mathsf{lb}}{\mathsf{gal}}$. Hence $\sigma_0(\cdot)$ is the constant-fnc S, as $(*)$ indeed says.

Induction. Fix a natnum N for which (45h) holds. Here, let y and σ denote y_{N+1} and σ_{N+1}. Our (45b) DE becomes

$$y'(t) = \overbrace{\mathrm{R} \cdot \sigma_N(t)}^{\text{Input}} - \overbrace{\mathrm{R} \cdot \sigma(t)}^{\text{Output}} .$$

Divide by U, the [constant] amount of water in each tank, to get FOLDE

$$45i: \quad \sigma'(t) + \mathbf{\Gamma} \cdot \sigma(t) = \mathbf{\Gamma} \cdot \sigma_N(t) .$$

As in (45c), FOLDE gives multiplier-fnc $M(t) := \mathsf{e}^{\mathbf{\Gamma} t}$. We wish to anti-diff product

$$P(t) := \mathsf{e}^{\mathbf{\Gamma} t} \cdot \mathbf{\Gamma} \cdot \sigma_N(t) \stackrel{\text{by (45h)}}{=} \mathrm{S}\mathbf{\Gamma} \cdot \mathsf{e}^{\mathbf{\Gamma} t} + \mathbf{\Gamma} \cdot f_N(t) .$$

Courtesy $(**)$, we can choose anti-deriv

$$Q(t) := \int^t P = \mathrm{S} \cdot \mathsf{e}^{\mathbf{\Gamma} t} + f_{N+1}(t) .$$

Adding the appropriate salinity constant α, then dividing by $M(t) = \mathsf{e}^{\mathbf{\Gamma} t}$, produces

$$\sigma_{N+1}(t) = \mathrm{S} + \frac{\alpha + f_{N+1}(t)}{\mathsf{e}^{\mathbf{\Gamma} t}} .$$

We've already checked that (45h) gives the correct value at $t = \boldsymbol{0}$, hence α must be $0 \frac{\mathsf{lb}}{\mathsf{gal}}$. The conclusion is that formula (45h) is correct at stage $N+1$. QED

§E Matrix-exp: A bit further

46.1: *If* $e^A e^B = e^B e^A$, *must* $A \leftrightarrows B$? *No.* The following example is from Prof. Howard Haber's [UC Santa Cruz] notes.

CEX. With $\tau \in \mathbb{C}$ *non-zero*, let $T := \begin{bmatrix} \tau & 1 \\ 0 & 0 \end{bmatrix}$. For $k = 1, 2, \ldots$, note $T^k = \begin{bmatrix} \tau^k & \tau^k/\tau \\ 0 & 0 \end{bmatrix}$. Furthermore,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = T^0 = \begin{bmatrix} \tau^0 & \tau^0/\tau \\ 0 & 0 \end{bmatrix} + C, \quad \textit{where } C := \begin{bmatrix} 0 & -1/\tau \\ 0 & 1 \end{bmatrix}.$$

Our defn $e^T \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} \frac{1}{k!} \cdot T^k$ results in

$$\begin{aligned} e^T &= \tfrac{1}{0!} \cdot C + \sum_{k=0}^{\infty} \tfrac{1}{k!} \cdot \begin{bmatrix} \tau^k & \tau^k/\tau \\ 0 & 0 \end{bmatrix} \\ &= C + \begin{bmatrix} e^\tau & e^\tau/\tau \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} e^\tau & [e^\tau - 1]/\tau \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

Let T_n be this matrix T when $\tau := n \cdot 2\pi\mathbf{i}$ and $n \in \mathbb{Z}$; since $e^\tau = 1$, *each* $\exp(T_n) = \mathbf{I}$, the identity matrix.

Last step. Matrix $A := \begin{bmatrix} 2\pi\mathbf{i} & 0 \\ 0 & 0 \end{bmatrix}$, is diagonal, hence

$$e^A = \begin{bmatrix} e^{2\pi\mathbf{i}} & \\ & e^0 \end{bmatrix} \stackrel{\text{note}}{=} \mathbf{I}.$$

With $B := T_1$, observe $A + B = T_2$. From above, then, $e^B = \mathbf{I} = e^{A+B}$. Consequently,

Each of $e^A e^B$, e^{A+B}, *and* $e^B e^A$, *equals* $\mathbf{I}$.

Yet A and B do not commute. For with $\tau := 2\pi\mathbf{i}$,

$$AB = \begin{bmatrix} \tau^2 & \tau \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} \tau^2 & 0 \\ 0 & 0 \end{bmatrix} = BA. \quad \blacklozenge$$

§F Intro to Calculus of Variations

(In progress.)

§Index for DiffyQ

Stated theorems are in the ToC. ***Applications** of theorems may appear in this index.*

That's All, Folks! –*Bugs Bunny*

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