Properties of determinant

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Abbreviations. These are defined below.
$\mathcal{MuL}$, multi-linear.
$\mathcal{AnS}$, anti-symmetric.
$\mathcal{Alt}$, alternating.

Intro. Below, all VSes are over a common field,

$$\mathsf{F} \quad = \quad (\mathsf{F}, +, 0, \cdot, 1)$$

In the defn below, I use "5" as a stand-in for N.

A map $g\colon \mathbf{V}_1 \times \cdots \times \mathbf{V}_5 \to \mathbf{X}$ is 5-***multi-linear*** [5-$\mathcal{MuL}$] if for all tuples $(\mathbf{u}_1, \ldots, \mathbf{u}_5) \in \mathbf{V}_1 \times \cdots \times \mathbf{V}_5$, the following 5 maps are each linear:

$$\begin{aligned} &g(\,\cdot\,, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5)\colon \mathbf{V}_1 \to \mathbf{X} \quad, \\ &g(\mathbf{u}_1, \,\cdot\,, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5)\colon \mathbf{V}_2 \to \mathbf{X} \quad, \\ &g(\mathbf{u}_1, \mathbf{u}_2, \,\cdot\,, \mathbf{u}_4, \mathbf{u}_5)\colon \mathbf{V}_3 \to \mathbf{X} \quad, \\ &g(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \,\cdot\,, \mathbf{u}_5)\colon \mathbf{V}_4 \to \mathbf{X} \quad, \\ &g(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \,\cdot\,)\colon \mathbf{V}_5 \to \mathbf{X}\,. \end{aligned}$$

Sometimes 5-$\mathcal{MuL}$ is called 5-***linear***, or just ***multi-linear*** if the 5 is understood. [We use ***bilinear*** rather than "2-*linear*", typically.]

When $\mathbf{V}_1 = \mathbf{V}_2 = \cdots = \mathbf{V}_5$. Map $h\colon \mathbf{V} \times \overset{5}{\cdots} \times \mathbf{V} \to \mathbf{X}$ is ***anti-symmetric*** [$\mathcal{AnS}$] if

†: *For all index-pairs $j < k$, switching the entries at indices j and k negates the h-value.*

E.g, for $j{=}2$ and $k{=}\,5$:

$$h(\mathbf{u}_1, \mathbf{u}_5, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_2) \;=\; -1 \cdot h(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5)\,.$$

A map $h\colon \mathbf{V} \times \overset{5}{\cdots} \times \mathbf{V} \to \mathbf{X}$ is ***alternating*** [$\mathcal{Alt}$] if for each tuple $\widehat{\mathbf{u}} = (\mathbf{u}_1, \ldots, \mathbf{u}_5)$:

*: *If there exists an index-pair $j < k$ with $\mathbf{u}_j = \mathbf{u}_k$, then $h(\widehat{\mathbf{u}}) = \mathbf{0}_\mathbf{X}$.*

1: AC-Alt Thm. *Consider a map $h\colon \mathbf{V} \times \overset{N}{\cdots} \times \mathbf{V} \to \mathbf{X}$. Then*

i: [$\mathcal{MuL}$ & $\mathcal{Alt}$] $\implies$ [$\mathcal{MuL}$ & $\mathcal{AnS}$].

ii: *When* $\mathrm{Char}(\mathsf{F}) \neq 2$ *then the converse holds:* [$\mathcal{MuL}$ & $\mathcal{AnS}$] $\implies$ [$\mathcal{MuL}$ & $\mathcal{Alt}$]. ◊

Proof of (i). WLOG, $N = 2$. For arbitrary $\mathbf{u},\mathbf{w} \in \mathbf{V}$, note

$$\begin{aligned} \mathbf{0}_\mathbf{X} &\overset{\text{by } \mathcal{Alt}}{=\!=\!=} h(\mathbf{u}{+}\mathbf{w},\, \mathbf{u}{+}\mathbf{w}) \\ &\overset{\text{by } \mathcal{MuL}}{=\!=\!=} h(\mathbf{u},\mathbf{u}) + h(\mathbf{u},\mathbf{w}) + h(\mathbf{w},\mathbf{u}) + h(\mathbf{w},\mathbf{w}) \\ &\overset{\mathcal{Alt}}{=\!=\!=} \mathbf{0}_\mathbf{X} + h(\mathbf{u},\mathbf{w}) + h(\mathbf{w},\mathbf{u}) + \mathbf{0}_\mathbf{X}\,. \end{aligned}$$

Hence $h(\mathbf{w},\mathbf{u}) \;=\; -h(\mathbf{u},\mathbf{w})$. ♦

Pf of (ii). $\mathcal{AnS}$ implies that $h(\mathbf{u},\mathbf{u}) \;=\; -h(\mathbf{u},\mathbf{u})$. Hence $[1+1]\cdot h(\mathbf{u},\mathbf{u}) \;=\; \mathbf{0}_\mathbf{X}$. Since $\mathrm{Char}(\mathsf{F}) \neq 2$, we may divide by $[1+1]$, obtaining the desired $h(\mathbf{u},\mathbf{u}) = \mathbf{0}_\mathbf{X}$. ♦

To verify that a map is $\mathcal{AnS}$, it suffices to check that

‡: *Switching adjacent entries, j and $j{+}1$, negates the value,*

as a transposition is a composition of *oddly many* adjacent-transpositions.

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