Cooling

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Autonomous-DiffyQ Example. Let $u=u(t)$ denote the temperature of a particular object at time t. Newton's law of cooling says there is a proportionality constant $\mathbf{k}$ [depending on the materials] so that

1a: $$\frac{du}{dt} = \mathbf{k} \cdot [u - \boldsymbol{A}],$$

where $\boldsymbol{A}$ denotes the ambient temperature.

Using abstract units ⓟ for temPerature, and ⓣ for Time, we have: $u, \boldsymbol{A} :: $ ⓟ and thus $\mathbf{k} :: 1/$ⓣ. Note that $\mathbf{k}$ will be negative, as the object's tmp will tend toward the ambient tmp.

Scenarios. If $\boldsymbol{A}$ is constant, then the situation is time-invariant, so the DE is autonomous.

Alternatively, $\boldsymbol{A}()$ and $u()$ might be linked U.Fncs; e.g, $u(t)$ is the [decreasing] temperature of the ice-cube, melting into your orange juice which is at [increasing] temperature $\boldsymbol{A}(t)$, both changing. Our (1a) is still an autonomous [time-invariant] DE, but now it relates two unknown-functions.

Alternatively, $\boldsymbol{A}=\boldsymbol{A}(t)$ might be a known fnc of time; e.g, we are in the desert, so $\boldsymbol{A}()$ is periodic with a 24-hour period. Here, (1a) is *not* an autonomous-DE. ☐

When $\boldsymbol{A}$ is constant. Here, (1a) is a FOLDE with constant coeff and target fncs. It general soln is

1b: $$u_\alpha(t) = \boldsymbol{A} + \alpha e^{\mathbf{k}t}.$$

Note that product $\mathbf{k}t$ is unitless ["dimensionless"] and so is $e^{\mathbf{k}t}$. Thus $\alpha ::$ ⓟ. ☐

Worked example. We remove a cake from a $320\,°$F oven. Three minutes later we measure the cake's temperature; it has cooled to $220\,°$F.

When will the cake be at an edible$^{♡1}$ tmp of $100\,°$F, given that the ambient tmp of the kitchen is $70\,°$F?

Step 1. *Replace all the quantities by meaningful letters.*

Below, I'll employ greek letters Δ and τ for time, and roman letters for temperature. With italic boldface $\boldsymbol{0}$ denoting 0min, define the following:

$$\Delta := \begin{bmatrix} \text{Time-change from} \\ \text{Oven to Measured} \end{bmatrix} = 3\mathsf{min}.$$

$$\tau := [\text{Edible time}].$$

$$\boldsymbol{A} := [\text{Ambient tmp}] = 70\,°\text{F}.$$

$$V := \begin{bmatrix} \text{Oven-tmp distance} \\ \text{to ambient} \end{bmatrix} = u(\boldsymbol{0}) - \boldsymbol{A}$$
$$= [320 - 70]\,°\text{F} = 250\,°\text{F}.$$

$$M := \begin{bmatrix} \text{Measured-tmp} \\ \text{dist. to ambient} \end{bmatrix} = u(\Delta) - \boldsymbol{A}$$
$$= [220 - 70]\,°\text{F} = 150\,°\text{F}.$$

$$E := \begin{bmatrix} \text{Edible-tmp dist.} \\ \text{to ambient} \end{bmatrix} = u(\tau) - \boldsymbol{A}$$
$$= [100 - 70]\,°\text{F} = 30\,°\text{F}.$$

So $\Delta, \tau ::$ ⓣ and $\boldsymbol{A}, V, M, E ::$ ⓟ.

Step 2. *Identify the unknowns, and solve for them.* Parameter $\mathbf{k}$, in the DE, and parameter α, in the soln, are unknown. [We'll get to time τ, later.]

Evaluating (1b) at $t = 0\mathsf{min}$ gives

$$\alpha = \alpha \cdot 1 = \alpha \cdot e^{\mathbf{k} \cdot \boldsymbol{0}} = u(\boldsymbol{0}) - \boldsymbol{A} \overset{\text{def}}{=\!=} V.$$

Hence

1b†: $$u(t) = \boldsymbol{A} + V e^{\mathbf{k}t}.$$

To solve for $\mathbf{k}$, note the tmp-drop over time Δ is

$$V - M = u(\boldsymbol{0}) - u(\Delta) = V \cdot [1 - e^{\mathbf{k}\Delta}].$$

Hence $e^{\mathbf{k}\Delta} = \frac{M}{V}$. Taking logs, then,

1c: $$\mathbf{k} = \frac{1}{\Delta} \cdot \log(M/V).$$

Plugging this in to (**??**†) gives

1d: $$u(t) = \boldsymbol{A} + V \cdot \left[\tfrac{M}{V}\right]^{t/\Delta}$$

for *the* soln to (1a) satisfying the init.conditions.

$^{♡1}$Apparently we can eat foods noticeably hotter than $100\,°$F.

Step 3. *Does the symbolic soln agree with known values?* Evaluating RhS$(1d)$ at $t=\mathbf{0}$ gives $\mathbf{A}+V$, which indeed equals $u(\mathbf{0})$, the LhS$(1d)$.

Evaluating RhS$(1d)$ at $t=\Delta$ delivers

$$\mathbf{A} + V\cdot\left[\tfrac{M}{V}\right] \;=\; \mathbf{A} + M \;\overset{\text{note}}{=\joinrel=\joinrel=}\; u(\Delta).$$

Vary the letters in your symbolic soln. Does the resulting answer vary the way the physics of the situation says it should?

E.g, lowering M [retaining all other values] means the cake lost more heat in those same $3\,\text{minutes}$; hence $u()$ should drop in temperature faster. And indeed, a decrease of M makes the smaller-than-one positive number $\tfrac{M}{V}$ even smaller, making its powers decrease faster.

Step 4. *Compute the desired quantities from the symbolic soln, then plug in the actual values.*

By its definition, time τ satisfies

$$\mathbf{A}+E \;\overset{\text{def}}{=\joinrel=\joinrel=}\; u(\tau) \;\overset{\text{by (1d)}}{=\joinrel=\joinrel=\joinrel=}\; \mathbf{A} \;+\; V\cdot\left[\tfrac{M}{V}\right]^{\tau/\Delta}.$$

Thus $\left[\tfrac{M}{V}\right]^{\tau/\Delta} = \tfrac{E}{V}$. Consequently,

1e: $$\tau \;=\; \Delta\cdot\frac{\log(E/V)}{\log(M/V)} \;\overset{\text{note}}{=\joinrel=\joinrel=}\; \Delta\cdot\frac{\log(V/E)}{\log(V/M)}.$$

This τ is a function of E. So, for a temperature-difference x, define

∗: $$\tau(x) \;:=\; \Delta\cdot\frac{\log(V/x)}{\log(V/M)}.$$

We expect...

...$\tau(V)$ to be $\mathbf{0}$, since the cake hasn't cooled at all.

...$\tau(M)$ to be Δ, since the cake is at the measured temperature.

...that the limit of $\tau(x)$, as x decreases toward $0\,°\text{F}$ [i.e, as edible-tmp decreases toward $\mathbf{A}$], to be $+\infty$; this, since the cake never achieves ambient temperature in finite time.

Reassuringly, all three of these expectations are fulfilled by RhS$(∗)$.

Finally, let's compute an approx. value for $\tau=\tau(E)$:

$$
\begin{aligned}
\tau &= 3\text{min}\cdot\frac{\log(250/30)}{\log(250/150)} = 3\text{min}\cdot\frac{\log(25/3)}{\log(5/3)} \\[2mm]
&\approx\quad 3\text{min}\cdot 4.15 \quad\approx\quad 13\text{min}.
\end{aligned}
$$