Two types of connected component

Jonathan L.F. King
University of Florida, Gainesville FL 32611-2082, USA
squash@ufl.edu
Webpage http://squash.1gainesville.com/
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ABSTRACT: In a topological space X we define two (sometimes) inequivalent notions of "component of a point x", $\mathbf{P}_x$ and $\mathbf{C}_x$.

The pseudo-component of a point

Use $\widehat{\mathcal{P}}$ for the family of X-clopen sets. Define

$$\mathcal{P}_x := \{E \subset X \mid E \text{ clopen and } E \ni x\}.$$

Let $\mathbf{P}_x := \bigcap(\mathcal{P}_x)$ be the closed set which is the intersection of the members of this collection. Now suppose a point $w \in \mathbf{P}_x$. Then $\mathcal{P}_x \subset \mathcal{P}_w$. Conversely

$$\begin{aligned} E \in \mathcal{P}_w &\implies E^c \in \widehat{\mathcal{P}} \smallsetminus \mathcal{P}_w \\ &\implies E^c \in \widehat{\mathcal{P}} \smallsetminus \mathcal{P}_x \\ &\implies E \in \mathcal{P}_x. \end{aligned}$$

Hence $\mathcal{P}_x = \mathcal{P}_w$. So if two intersections $\mathbf{P}_x$ and $\mathbf{P}_y$ share a point, w, then $\mathbf{P}_x = \mathbf{P}_w = \mathbf{P}_y$. We conclude

1: $\{\mathbf{P}_x \mid x \in X\}$ *is a partition of* X *into closed subsets, with* $x \in \mathbf{P}_x$.

As shown later, an $\mathbf{P}_x$ *need not* be connected; this is a good reason not to call it a "connected" component! This $\mathbf{P}_x$ is called the ***pseudo-component of*** x.

(Maximal) connected component of a point

Let $\mathcal{C}_x := \{K \mid K \text{ connected and } K \ni x\}$. Let $\mathbf{C}_x := \bigcup(\mathcal{C}_x)$ be the union of this collection.

2: Lemma. *$\mathbf{C}_x$ is a connected subset of X.* ♢

Pf. Fix a set $V \subset \mathbf{C}_x$ which is clopen in the relative topology of $\mathbf{C}_x$ and which owns x. Thus

$$V = \mathbf{C}_x \cap A \quad \text{and} \quad \mathbf{C}_x \smallsetminus V = \mathbf{C}_x \cap B,$$

for some X-open subsets A,$B \subset X$; these open sets need not be disjoint.

Consider a $K \in \mathcal{C}_x$. In the relative topology of K, the intersections $K \cap A$ and $K \cap B$ are open and (since K is inside $\mathbf{C}_x$) are disjoint. As $A \cap K$ is clopen in K it *equals* K, as this latter is connected and intersects A (at x). Hence $V \supset K$. Holding for all K, this says that $\boxed{V \text{ equals } \mathbf{C}_x}$. ♦

Partitioning follows by observing that

$$w \in \mathbf{C}_x \implies \mathcal{C}_x \cap \mathcal{C}_w \neq \varnothing \implies x \in \mathbf{C}_w.$$

Thus $\mathbf{C}_x = \mathbf{C}_w$. As above, this yields that if $\mathbf{C}_x$ and $\mathbf{C}_y$ intersect, then $\mathbf{C}_x = \mathbf{C}_y$. Consequently

3: $\{\mathbf{C}_x \mid x \in X\}$ *is a partition of* X *into connected sets, with* $x \in \mathbf{C}_x$.

Implications and examples

If E is a clopen set intersecting a connected set K, then $E \supset K$ as argued above. Hence $\mathbf{P}_x \supset \mathbf{C}_x$ always.

The connected component can be smaller than the pseudo-component. Consider the plane $\mathbb{R}^2$.

For $v \in \mathbb{R}$, let L_v denote the line segment $[0,1]\times\{v\}$ in the plane. For an arbitrary subset $S \subset L_0$ define

$$\Omega := S \sqcup \bigsqcup_{n=1}^{\infty} L_{1/n}$$

endowed with the induced metric.

Pseudo-comp. Fix an $x \in S$ and $\mathbb{R}^2$-open ball $U \ni x$. For all large n, then, U intersects $L_{1/n}$.

Each $L_{1/n}$ is connected, so U includes some union $\bigcup_{n=?}^{\infty} L_{1/n}$. Supposing now that $U \cap \Omega$ is Ω-clopen, note that $U \cap \Omega$ must include the Ω-closure of this union; so $U \cap \Omega \supset S$. Thus

In metric space Ω: For each $x \in S$, its intersection-set $\mathbf{P}_x$ includes S. Hence S lies inside a single pseudo-component.

It follows that $\boxed{S \text{ \underline{is} an } \Omega\text{–pseudo-component}}$, since each $L_{1/n}$ is. □

Connected-comp. Conversely, suppose $K \subset \Omega$ is a connected. If it touches a particular $L_{1/n}$ then it is included in it, since $L_{1/n}$ is clopen in Ω. So a connected set $K \subset \Omega$ which touches S, is included in S. Thus

For each $x \in S$: Its Ω-component $\mathbf{C}_x$ equals its S-component, where we view S as a metric space.

In particular, if S is totally disconnected then each $\mathbf{C}_x = \{x\}$, whereas $\mathbf{P}_x = S$. A striking example is the two-point set $S := \{(0,0),(1,0)\}$. □

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