**Notes in the key of $\mathbb{C}$**

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*Whoa!* Not figured out, yet.

*Frobenius Number.* Posint set $\boldsymbol{S} = \{s_1, s_2, \ldots, s_J\}$ is coprime, i.e $\mathrm{GCD}(\boldsymbol{S}) = 1$. The set of non-negative linear combinations (non-negative ***lincombs***) is

$$\mathsf{NNLC}(\boldsymbol{S}) \;:=\; \Big\{\sum_{j=1}^{J} u_j s_j \;\Big|\; \text{Each } u_j \in \mathbb{N}\Big\};$$

the ***S-representable numbers***. The coprimeness of $\boldsymbol{S}$ easily implies that all large integers are representable. The largest *non*-representable number is $\mathrm{Frob}(\boldsymbol{S})$, the ***Frobenius number*** of $\boldsymbol{S}$. □

**1.1: Lemma.** *$\mathrm{Frob}(A, B) = AB - [A+B]$ whenever $A \perp B$ are posints.* ♢

*Pf.* For $\alpha = 0, 1, \ldots, A-1$, sets $F_\alpha := \alpha B + A\mathbb{Z}$ are pairwise disjoint, and $H_\alpha := \alpha B + A\mathbb{N}$ are pairwise disjoint. So $\bigsqcup_\alpha F_\alpha = \mathbb{Z}$ and $\bigsqcup_\alpha H_\alpha = \mathsf{NNLC}(A, B)$.

The most-positive element in $F_\alpha \smallsetminus H_\alpha$ is $[\alpha B] - A$; so the most-positive of those is when $\alpha = A-1$.

## FiCA: 10.4 Coin-Exchange Problem

Henceforth:

*Posints $A, B$ are a fixed coprime pair.*

Problem **10.4** [FiCA P.144] asks: *Given natnum "target" $\boldsymbol{\tau}$*, compute $\mathcal{N}(\boldsymbol{\tau})$, the # of $(A, B)$-representations of $\boldsymbol{\tau}$. I.e, $\mathcal{N}(\boldsymbol{\tau})$ is the number of natnum-pairs $(u, v)$ with $uA + vB = \boldsymbol{\tau}$.

[For example, we expect $\mathcal{N}(0) = \mathcal{N}(A) = \mathcal{N}(B) = 1$ and $\mathcal{N}(\mathrm{Frob}(A, B)) = 0$.]

**Idea.** The coeff of $z^{\boldsymbol{\tau}}$ in product $[1 - z^A][1 - z^B]$, i.e in

$$[1 + z^A + z^{2A} + z^{3A} + \ldots] \cdot [1 + z^B + z^{2B} + z^{3B} + \ldots]$$

is $\mathcal{N}(\boldsymbol{\tau})$. Defining denominator polynomial

**1.2:**
$$P(z) \;:=\; [1 - z^A][1 - z^B] z^{\boldsymbol{\tau}+1}, \quad \textit{then}$$
$$\operatorname*{Res}_{z=0}(f) \;=\; \mathcal{N}(\boldsymbol{\tau}), \quad \textit{where} \quad f \;:=\; 1/P.$$

**Residues.** With $\boldsymbol{\alpha} := \mathrm{e}^{\frac{2\pi \mathbf{i}}{A}}$, let $\mathcal{A} := \{\boldsymbol{\alpha}^j\}_{j=1}^{A-1}$; the set of $A^{th}$-th roots-of-unity, excluding 1. Define $\boldsymbol{\beta}$ and $\mathcal{B}$, similarly.

As $A \perp B$, intersection $\mathcal{A} \cap \mathcal{B}$ is empty. Union $\mathcal{A} \cup \mathcal{B}$ comprises the $f$-simple-poles. Value $\boldsymbol{\alpha}^0 = 1 = \boldsymbol{\beta}^0$ is an order-2 pole. At the origin, the pole has order-$[\boldsymbol{\tau}+1]$.

Below, use $\langle \cdot \rangle_B$ for mod-$B$ residue, e.g, $\langle 3B + 7 \rangle_B$ equals $\langle 7 \rangle_B$.

**1.3: Prop.** *Fix $j \in [1 .. B)$. At $\boldsymbol{b} := \boldsymbol{\beta}^j$,*

$$\operatorname*{Res}_{z=\boldsymbol{b}}(f) \;=\; \frac{-1/B}{[1 - \boldsymbol{\beta}^{\langle jA \rangle_B}] \cdot \boldsymbol{\beta}^{\langle j\boldsymbol{\tau} \rangle_B}}. \qquad ♢$$

*Pf.* Note first that

$$\begin{aligned}[1 - z^A] \cdot z^{\boldsymbol{\tau}+1} \rfloor_{z=\boldsymbol{b}} &= [1 - \boldsymbol{\beta}^{\langle jA \rangle_B}] \cdot \boldsymbol{\beta}^{\langle j\boldsymbol{\tau}+j \rangle_B} \\ &= \boldsymbol{\beta}^{\langle j\boldsymbol{\tau}+j \rangle_B} - \boldsymbol{\beta}^{\langle j[A+\boldsymbol{\tau}]+j \rangle_B}\end{aligned}$$

By l'Hôpital,

$$\lim_{z \to \boldsymbol{b}} \frac{z - \boldsymbol{b}}{1 - z^B} = \lim_{z \to \boldsymbol{b}} \frac{1}{[-B] \cdot z^{B-1}} = \frac{1}{[-B] \cdot \boldsymbol{\beta}^{jB-j}} = \frac{\boldsymbol{\beta}^j}{[-B]}.$$

Consequently, $\mathrm{Res}_{z=\boldsymbol{b}}(f)$ equals

$$\frac{\boldsymbol{\beta}^j}{[-B]} \cdot \frac{1}{[1 - \boldsymbol{\beta}^{\langle jA \rangle_B}] \cdot \boldsymbol{\beta}^{\langle j\boldsymbol{\tau}+j \rangle_B}} \qquad ♦$$

**Summing residues.** With $\mathsf{C}_r := \mathrm{Sph}_r(0)$, our $f$ is *outlandish* w.r.t $(\mathsf{C}_1, \mathsf{C}_2, \ldots)$, since the degree of the denominator is $A + B + \boldsymbol{\tau} + 1 \geq 3$. Hence the sum of all the $f$-residues is zero.