A C^∞ function which is not everywhere analytic

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The goal of this note is to produce a $\mathbf{C}^\infty$-function $\widehat{\mathbf{F}}:\mathbb{R}\circlearrowleft$ whose Taylor series (centered at zero) converges to a *different* fnc —namely, to the zero-function.

On $U := \mathbb{R}\setminus\{0\}$, the following fnc $\mathbf{F}()$ is strictly positive; thus it differs from the zero-fnc on all of U.

$$\text{1a:}\quad \mathbf{F}(x) := \mathsf{e}^{-1/x^2}\,; \quad \widehat{\mathbf{F}}(x) := \begin{cases} 0\,, & \text{if } x = 0 \\ \mathbf{F}(x)\,, & \text{if } x \neq 0 \end{cases}.$$

[We'll soon see that $\widehat{\mathbf{F}}$ is ∞ly *Flat* at the origin.]

Generalizing, a degree-D poly $P(z) := \sum_{j=0}^{\mathrm{D}} \mathrm{C}_j z^j$ [each C_j is a number] defines a function $\mathbf{V}_P$ by

$$\text{1b:}\qquad \mathbf{V}_P(x) := \begin{cases} 0\,, & \text{if } x = 0 \\ \mathbf{F}(x)\cdot P(\tfrac{1}{x})\,, & \text{if } x \neq 0 \end{cases}.$$

[So the $\widehat{\mathbf{F}}$ from (1a) is $\mathbf{V}_1$.] It may not be evident that $\mathbf{V}_P$ is differentiable at 0. But certainly at *non-zero* x, we can use the Product rule to compute as follows:

$$\begin{aligned} [\mathbf{V}_P]'(x) &= \mathbf{F}'(x)P(\tfrac{1}{x}) + \mathbf{F}(x)\cdot P'(\tfrac{1}{x})\cdot\tfrac{-1}{x^2} \\ &= [\mathbf{F}(x)\tfrac{2}{x^3}]P(\tfrac{1}{x}) - \mathbf{F}(x)\cdot\tfrac{1}{x^2}P'(\tfrac{1}{x}) \\ &= \mathbf{F}(x)\cdot\Big[2[\tfrac{1}{x}]^3 P(\tfrac{1}{x}) - [\tfrac{1}{x}]^2 P'(\tfrac{1}{x})\Big]. \end{aligned}$$

This suggests defining an operation on polynomials. Given a poly P, define a new poly, $\widetilde{P}$, by

$$\text{1c:}\qquad \widetilde{P}(z) := 2z^3 P(z) - z^2 P'(z)\,.$$

The computation above showed, for each $x \neq 0$, that $[\mathbf{V}_P]'(x)$ equals $\mathbf{V}_{\widetilde{P}}(x)$. Now let's finish the job.

2a: Theorem. *For each polynomial P, the function $\mathbf{V}_P$ from (1b) is everywhere differentiable. Moreover, $[\mathbf{V}_P]' = \mathbf{V}_{\widetilde{P}}$.* ◊

Proof. What is left to show is that $[\mathbf{V}_P]'(0)$ equals 0. Happily, the definition of derivative tells us that

$$\begin{aligned} [\mathbf{V}_P]'(0) &\stackrel{\text{def}}{=} \lim_{x\to 0} \frac{\mathbf{V}_P(x) - \mathbf{V}_P(0)}{x - 0} \\ &= \lim_{x\to 0} \frac{\mathbf{F}(x)P(\frac{1}{x})}{x} \\ &= \lim_{x\to 0} \mathbf{F}(x)\cdot\tfrac{1}{x}P(\tfrac{1}{x})\,. \end{aligned}$$

By Prop'n 3c [proved further below] this latter equals zero We'll apply (3c) by defining the $Q()$ of that proposition to be $Q(z) := z\cdot P(z)$. ♦

2b: Corollary. *Given an arbitrary polynomial P, define a sequence of polys by $P_0 := P$ and $P_{n+1} := \widetilde{P_n}$. Then $\mathbf{V}_P$ is ∞ly differentiable, and its n^{th} derivative satisfies*

$$[\mathbf{V}_P]^{(n)} = \mathbf{V}_{P_n}\,,$$

for each $n = 0, 1, 2, \ldots$. In particular,

$$\widehat{\mathbf{F}}^{(n)} = \mathbf{V}_{\mathsf{p}_n}\,,$$

where $\mathsf{p}_0() := 1$ and $\mathsf{p}_{n+1} := \widetilde{\mathsf{p}_n}$. ◊

3a: Lemma. *For each integer $N \geq 0$, the limit*

$$\ell_N := \lim_{u\nearrow\infty} \frac{u^N}{\mathsf{e}^{[u^2]}}$$

exists, and equals zero. ◊

Proof. Certainly ℓ_0 is zero. I now induct on $N\in\mathbb{Z}_+$. By L'Hôpital's rule,

$$\begin{aligned} \ell_N &= \lim_{u\nearrow\infty} \frac{N u^{N-1}}{2u\cdot\mathsf{e}^{[u^2]}} \\ &= \tfrac{N}{2}\Big[\lim_{u\nearrow\infty}\tfrac{1}{u}\Big]\Big[\lim_{u\nearrow\infty}\tfrac{u^{N-1}}{\mathsf{e}^{[u^2]}}\Big], \quad \textit{since the limit of a product is the product of the limits, if both limits exist in } \mathbb{R}, \\ &= \tfrac{N}{2}\cdot 0\cdot\ell_{N-1} = \tfrac{N}{2}\cdot 0\cdot 0 = 0\,, \quad \text{by induction.} \end{aligned}$$ ♦

Rem. This lemma implies, by letting $u := 1/x$, that

$$\lim_{x\searrow 0} \exp(-\tfrac{1}{x^2})\cdot\tfrac{1}{x^N} = \lim_{u\nearrow\infty} \mathsf{e}^{-[u^2]}\cdot u^N = 0\,.$$

Indeed, we conclude that this holds for the two-sided limit,

$$\text{3b:}\qquad \lim_{x\to 0} \mathbf{F}(x)\cdot\tfrac{1}{x^N} = 0\,,$$

since $\big|\mathbf{F}(x)\tfrac{1}{x^N}\big|$ equals $\mathbf{F}(x)\cdot\tfrac{1}{|x|^N}$. □

3c: Polynomial proposition. *For an arbitrary polynomial Q, necessarily*

$$*:\qquad \lim_{x\to 0} \mathbf{F}(x)\cdot Q(\tfrac{1}{x}) \quad = \quad 0\,. \qquad \diamondsuit$$

Proof. Write $Q(z) = \mathrm{C}_0 + \mathrm{C}_1 z + \cdots + \mathrm{C}_\mathrm{D} z^\mathrm{D}$. Then

$$\begin{aligned}\lim_{x\to 0} \mathbf{F}(x)\,Q(\tfrac{1}{x}) &= \sum_{j=0}^{\mathrm{D}} \mathrm{C}_j \cdot \Big[\lim_{x\to 0} \mathbf{F}(x)\tfrac{1}{x^j}\Big] \\ &= \sum_{j=0}^{\mathrm{D}} \mathrm{C}_j \cdot 0\,, \quad \text{by (3b)}.\end{aligned}$$

And this last sum equals zero, as desired. ♦

For the curious. The n^{th}-derivative of $\widehat{\mathbf{F}}$ is $\mathbf{V}_{P_n}$, where update rule (1c) gives

n	$P_n(z)$
0	1
1	$2z^3$
2	$4z^6 - 6z^4$
3	$8z^9 - 36z^7 + 24z^5$
4	$16z^{12} - 144z^{10} + 300z^8 - 120z^6$

h diff'able but h' discts. Consider fnc $h:\mathbb{R}\to\mathbb{R}$ by

$$h(x) \;:=\; \begin{Bmatrix} 0\,, & \text{if } x=0 \\ x^2\cos(\frac{1}{x^{100}})\,, & \text{if } x\neq 0 \end{Bmatrix}.$$

Is h diff'able at the origin? Claim: $h'(0)=\mathbf{0}$. Difference-quotient $\frac{h(t)-h(0)}{t-0} \overset{\text{note}}{=\!=} \frac{h(t)}{t} = t\cdot\cos(\frac{1}{t^{100}})$. Since

4: $$\left|\frac{h(t)-h(0)}{t-0} - \mathbf{0}\right| \;\le\; |t|\cdot 1 \;=\; |t|$$

goes to zero as $t\to 0$, we've proved $\boxed{h'(0)=\mathbf{0}}$. Since h is $\mathbf{C}^\infty$ off the origin, h is diff'able everywhere. Easily,

$$\Big[\limsup_{t\to 0} h(t)\Big] = +\infty \quad \textit{and} \quad \Big[\liminf_{t\to 0} h(t)\Big] = -\infty\,.$$

Our poor h is explosively discontinuous at the origin.

Vectorspace dervative. *The below is stated for real-vectorspaces. For complex VSes, replace $\mathbb{R}$ by $\mathbb{C}$, and replace 'real-linear' by 'complex-linear'.*

Consider map $f:\mathbb{R}^N\to\mathbb{R}^J$ [*with* $N,J\in\mathbb{Z}_+$] and point $\mathbf{p}\in\mathbb{R}^N$. For nearby point $\mathbf{p}+v$, abbreviate the change in f by

$$^{\Delta}f(v) \;:=\; f(\mathbf{p}+v) \;-\; f(\mathbf{p})\,.$$

Our f is ***differentiable*** at $\mathbf{p}$ if there is a $\mathbb{R}$-*linear* map $L:\mathbb{R}^N\to\mathbb{R}^J$ with

$$\lim_{v\to\mathbf{0}} \frac{\big\|{}^{\Delta}f(v) \;-\; L(v)\big\|_J}{\|v\|_N} \;=\; \mathbf{0}\,.$$

In general, a map $f:(\mathbf{A},\mathbf{0}_\mathbf{A},\,\|\cdot\|_\mathbf{A})\to(\mathbf{B},\mathbf{0}_\mathbf{B},\,\|\cdot\|_\mathbf{B})$ between two normed VSes, [mapping *amber* $\mathbf{A}$ to *blue* $\mathbf{B}$] is diff'able at $\mathbf{p}\in\mathbf{A}$ if

$$\lim_{v\to\mathbf{0}_\mathbf{A}} \frac{\big\|{}^{\Delta}f(v) \;-\; L(v)\big\|_\mathbf{B}}{\|v\|_\mathbf{A}} \;=\; \mathbf{0}_\mathbf{B}\,,$$

for some linear $L:\mathbf{A}\to\mathbf{B}$. □

$\mathbb{R}$-analyticity is not sealed under uniform limits

We now obtain the preceding fnc $\widehat{\mathbf{F}}$ as a uniform-limit (on $\mathbb{R}$) of "*flatish*" functions[♡1] $\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3, \ldots$, where

$$\mathbf{f}_N(x) \quad := \quad \exp\Bigl(\frac{-N}{1+Nx^2}\Bigr).$$

Rewriting $\mathbf{f}_N(x) = 1/\exp(\frac{1}{[1/N]+x^2})$ shows that, pointwise, $\mathbf{f}_1 \geq \mathbf{f}_2 \geq \mathbf{f}_3 \geq \cdots \geq 0$. Of course, the completeness of the reals implies that the pointwise limit $[\lim_{n\to\infty} \mathbf{f}_n]$ exists; evidently, this limit is the $\widehat{\mathbf{F}}$ from (1a). But this even stronger result holds:

5: Theorem. *On $\mathbb{R}$ these $\mathbf{f}_n \xrightarrow[n\to\infty]{\text{uniformly}} \widehat{\mathbf{F}}$.* ◇

Pf. Use $\|\cdot\|$ for the sup-norm on $\mathbb{R}$. Letting $J{=}J(N)$ denote the cube-root of posint N, my goal is

6: $$\|\mathbf{f}_N - \widehat{\mathbf{F}}\| \quad \leq \quad \mathrm{Max}\Bigl\{[\mathrm{e}^{1/J}] - 1,\ \frac{1}{\mathrm{e}^{J/2}}\Bigr\}.$$

Both terms in Max{} go to zero as $J\nearrow\infty$, so this will establish (5).

Note $\mathbf{f}_N(0) - \widehat{\mathbf{F}}(0) = \mathrm{e}^{-N} \leq \mathrm{e}^{-J/2}$, which is the righthand term of Max{}. So ISTShow for each *nonzero* x that $\mathbf{f}_N(x) - \widehat{\mathbf{F}}(x) \leq \mathrm{RhS}(6)$.

The substitution $z := x^2 \overset{\text{note}}{>} 0$ reduces our task to establishing that

6′: $$\Bigl[\sup_{z\in\mathbb{R}_+} [\mathrm{e}^{\frac{-N}{1+Nz}} - \mathrm{e}^{\frac{-1}{z}}]\Bigr] \overset{?}{\leq} \mathrm{Max}\Bigl\{[\mathrm{e}^{1/J}] - 1,\ \mathrm{e}^{-J/2}\Bigr\}.$$

To this end, let $\mathbf{s}(z) := [\mathrm{e}^{\frac{-N}{1+Nz}} - \mathrm{e}^{\frac{-1}{z}}]$. Fix a positive z and perceive[♡2] that

7: $$\mathbf{s}(z) \ \leq \ \mathrm{e}^{\frac{-N}{1+Nz}} \ =: \ \mathcal{I}(z).$$

We get an alternate inequality, (8), by factoring,

$$\mathbf{s}(z) \quad = \quad \mathrm{e}^{\frac{-1}{z}} \cdot \bigl[\mathrm{e}^{\frac{1}{z[1+Nz]}} - 1\bigr].$$

But $\mathrm{e}^{\frac{-1}{z}} \leq 1$, so $\mathbf{s}(z) \leq \mathrm{e}^{\frac{1}{z[1+Nz]}} - 1$. Reducing the denominator $z[1+Nz]$ to Nz^2 gives

8: $$\mathbf{s}(z) \ \leq \ \bigl[\mathrm{e}^{1/Nz^2}\bigr] - 1 \ =: \ \mathcal{D}(z).$$

[♡1]Each $\mathbf{f}_n$ is *real-analytic*, but is not complex-plane-analytic. [Not *entire*.] The rational fnc $\mathbf{r}_n(x) := \frac{-n}{1+nx^2}$ has poles at $\pm\mathsf{p}_n$, where $\mathsf{p}_n := \mathbf{i}/\sqrt{n}$. While p_n is just a *pole* of $\mathbf{r}_n$, this p_n is an *essential singularity* of our $\mathbf{f}_n \overset{\text{def}}{=\!=} \exp\circ\mathbf{r}_n$. So p_n is a trouble-point of $\mathbf{f}_n$. Note $\lim_n \mathsf{p}_n = 0$ and, unsurprisingly, zero is the trouble-point for $\widehat{\mathbf{F}} \overset{\text{note}}{=\!=} \lim_n \mathbf{f}_n$.

[♡2]Can "perceive" really be used in the imperative?

Maximizing over two $\mathbb{R}_+$-intervals. This $\mathcal{D}(z)$ is a decreasing fnc of $z \in \mathbb{R}_+$. For each z in interval $[\frac{1}{J}, \infty)$, then, $\mathcal{D}(z) \leq \mathcal{D}(\frac{1}{J})$. Since $N \cdot [\frac{1}{J}]^2 = J$,

8′: $$\mathbf{s}(z) \quad \leq \quad [\mathrm{e}^{1/J}] - 1.$$

The $\mathcal{I}(z)$ from (7) is an increasing fnc of $z \in \mathbb{R}_+$. For each $z \in (0, \frac{1}{J}]$, then, $\mathcal{I}(z) \leq \mathcal{I}(\frac{1}{J})$. Thus

$$\begin{aligned}\mathbf{s}(z) &\leq \mathrm{e}^{\frac{-J^3}{1+J^2}} \\ &\leq \mathrm{e}^{\frac{-J^3}{J^2+J^2}}, \text{ since } J \geq 1 \text{ because } N \geq 1, \\ &\leq \mathrm{e}^{-J/2}.\end{aligned}$$

This, together with (8′), implies (6′). ♦

A $\mathbf{C}^\infty$ function on $\mathbb{R}$ with compact support

Taking the $\mathbf{F}$ from (1a) and zero-ing it out on the non-positive real axis, *halves* the fnc, giving

9: $$\mathbf{h}(x) \ := \ \begin{Bmatrix} 0, & \text{if } x \leq 0 \\ \exp(-\frac{1}{x^2}), & \text{if } x > 0 \end{Bmatrix}$$

This is a $\mathbf{C}^\infty_{\mathbb{R}}$-function whose n^{th}-derivative is

$$\mathbf{h}^{(n)}(x) \ := \ \begin{Bmatrix} 0, & \text{if } x \leq 0 \\ \exp(-\frac{1}{x^2}) \cdot R_n(\frac{1}{x}), & \text{if } x > 0 \end{Bmatrix},$$

where polynomial R_n is from (2b).

For $k = 1, 2, \ldots$, define a "bump fnc" or "test fnc"

$$\mathbf{b}_k(x) \quad := \quad \mathbf{h}(\tfrac{1}{k} + x) \cdot \mathbf{h}(\tfrac{1}{k} - x).$$

This $\mathbf{C}^\infty_{\mathbb{R}}$-function has two points of non-analyticity; the points $\pm\frac{1}{k}$. The support of $\mathbf{b}_k$ is open interval

$$\mathrm{Supp}(\mathbf{b}_k) \ = \ (-\tfrac{1}{k}, \tfrac{1}{k}),$$

which is bounded. Hence one possible definition of the Dirac-delta is the distributional-limit $\lim_{k\to\infty} \mathbf{b}_k$.