

## Chromatic polynomial of a graph

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### Polynomial preliminaries.

A poly(nomial) such as  $f(x) = 8x^2 - 7x + 2$  can be written as  $8\mathbf{e}_2 - 7\mathbf{e}_1 + 2\mathbf{e}_0$ , where  $\mathbf{e}_j(x) := x^j$ . As a linear combination,

$$\dagger: \quad f = \sum_{j=0}^2 \alpha_j \mathbf{e}_j,$$

where  $\vec{\alpha} := (\alpha_0, \alpha_1, \alpha_2) = (2, -7, 8)$ . The product of  $f$  with a degree-3 polynomial  $g = \sum_{k=0}^3 \beta_k \mathbf{e}_k$  is

$$f \cdot g = \sum_{n=0}^5 \gamma_n \mathbf{e}_n,$$

where, summing over natnum-pairs  $(j, k)$ , each

$$1a: \quad \gamma_n = \sum_{(j,k): j+k=n} [\alpha_j \cdot \beta_k].$$

This  $\vec{\gamma} = (\gamma_0, \dots, \gamma_5)$  is  $\vec{\alpha} \otimes \vec{\beta}$ , the *convolution* of  $\vec{\alpha}$  with  $\vec{\beta}$ . An alternative notation to  $(\dagger)$  is

$$\ddagger: \quad f = \sum_{j=0}^{\infty} \alpha_j \mathbf{e}_j,$$

where  $\alpha_j = 0$  when  $j > 2$ . Such an  $\vec{\alpha} = (\alpha_0, \alpha_1, \dots)$  is an *eventually-zero* sequence.

We'll also need the *falling-factorial* polynomials,

$$[\![x \downarrow N]\!] := x \cdot [x-1] \cdot [x-2] \cdots [x-[N-1]]$$

(a product of  $N$  many terms), where  $N \in \mathbb{N}$ . How to write a poly such as  $F(x) := 2x^3 - x^2 - 3x + 3$  in terms of falling factorials? The coeff of  $x^3$  is 2, so we subtract,

$$F(x) - 2 \cdot [\![x \downarrow 3]\!] = 5x^2 - 7x + 3.$$

From this we subtract  $5 \cdot [\![x \downarrow 2]\!]$ , producing

$$-2x + 3.$$

From *this* we subtract  $-2 \cdot [\![x \downarrow 1]\!]$ , yielding 3. From 3 we subtract  $3 \cdot [\![x \downarrow 0]\!]$ , ending with zero. Thus

$$F(x) = 3 \cdot [\![x \downarrow 0]\!] + -2 \cdot [\![x \downarrow 1]\!] + 5 \cdot [\![x \downarrow 2]\!] + 2 \cdot [\![x \downarrow 3]\!].$$

Letting  $\mathbf{t}_n(x) := [\![x \downarrow n]\!]$ , we have equality

$$1b: \quad \begin{aligned} F &= 3\mathbf{e}_0 - 3\mathbf{e}_1 - 1\mathbf{e}_2 + 2\mathbf{e}_3 \\ &= 3\mathbf{t}_0 - 2\mathbf{t}_1 + 5\mathbf{t}_2 + 2\mathbf{t}_3. \end{aligned}$$

We have done a change-of-basis computation, from basis  $(\mathbf{e}_j)_{j=0}^{\infty}$  to the  $(\mathbf{t}_n)_{n=0}^{\infty}$  basis.  $\square$

*Chromatic form.* Looking ahead, consider a monic intpoly  $\mathcal{P}()$  with  $R \in \mathbb{N}$  many [not necessarily distinct] integer-roots  $Z_1, Z_2, \dots, Z_R$ . Writing  $\mathcal{P}(x)$  in *chromatic form* means writing

$$1c: \quad \mathcal{P}(x) = \left[ \prod_{j=1}^R [x - Z_j] \right] \cdot f(x),$$

where [either “ $\cdot f(x)$ ” is absent or]  $f$  is a monic intpoly with no integral roots. Indeed, courtesy the Gauss Lemma for polynomials, our  $f$  has *no rational roots*.

When  $\mathcal{P}$  is a chromatic polynomial, its integer-root part  $\prod_j [x - Z_j]$  should be written in form

$$*: \quad x^{e_0} \cdot [x-1]^{e_1} \cdot [x-2]^{e_2} \cdots [x-[K-1]]^{e_{K-1}},$$

often mixed with falling-factorials, e.g

$$**: \quad [x \downarrow 2]^5 \cdot [x \downarrow 4] \cdot [x-1]^7.$$

See (4d), the Chromatic-polynomial Corollary.  $\square$

**Graph terminology.** Use  $\mathcal{K}_n$  for the *complete graph* on  $n$  vertices, and  $\mathcal{K}_{j,k}$  for the *complete  $(j,k)$ -bipartite graph*. Use  $\mathcal{C}_n$  for the *cyclic graph* with  $n$  vertices and  $n$  edges [ $\mathcal{C}_1$  has a single self-edge;  $\mathcal{C}_2$  has 2 edges between 2 vertices; for  $n \geq 3$ , our  $\mathcal{C}_n$  is a simple graph]. Use  $\mathcal{E}mp_n$  for the  $n$ -vertex graph with *no* edges; an “Empty graph”.

**Coloring.** Consider  $\mathcal{G} = (\mathbb{V}, \mathbb{E})$ , a finite graph, [loops, multiple-edges ok] with  $N := |\mathbb{V}|$ , and  $L := |\mathbb{E}|$ . Use  $\mathbf{c}(\mathcal{G})$  for the # of *connected components* of  $\mathcal{G}$ .

For  $k \in \mathbb{N}$ , a “ $k$ -coloring” of  $\mathcal{G}$  means to assign a “color”, an element of  $[1 \dots k]$ , to each vertex, so that no two *neighbors* [the end-vertices of an edge] have the same color. A  $k$ -coloring is *full* if it uses all  $k$  colors. The *chromatic number* of  $\mathcal{G}$ , written  $\chi(\mathcal{G})$ , is the minimum number of colors needed. I.e, it is the unique  $k$  so that there *is* a  $k$ -coloring of  $\mathcal{G}$ , and every  $k$ -coloring of  $\mathcal{G}$  is full.

For  $k = 0, 1, 2, \dots$ , let  $\mathcal{P}_{\mathcal{G}}(k)$  be the *number* of  $k$ -colorings of  $\mathcal{G}$ . Note  $\mathcal{P}_{\mathcal{E}mp_0}(k) = 1$  since the *void graph* [no vertices] has exactly one  $k$ -coloring. Chromatic number  $\chi(\mathcal{G})$  is the smallest natnum  $k$  for which  $\mathcal{P}_{\mathcal{G}}(k)$  is positive.  $\heartsuit_1$

$\heartsuit_1$  If  $\mathcal{G}$  has a loop, then  $\mathcal{P}_{\mathcal{G}}() \equiv 0$  and  $\chi(\mathcal{G}) := \infty$ .

If  $\mathcal{G}$  has multiple edges between vertices  $\mathbf{u}, \mathbf{v}$ , then replacing them by a single edge will not change the chromatic poly/number. Without announcement we will do this; effectively, we compute chromatic polys only of *simple* graphs.

**Deletion-contraction.** Consider an  $N$ -vertex  $G$  and edge  $\alpha \in \mathbb{E}$ . Let  $G \setminus \{\alpha\}$  mean to *delete* edge  $\alpha$ ; no vertices are removed, so  $G \setminus \{\alpha\}$  still has  $N$  vertices. In contrast, use  $G/\alpha$  to mean the graph with  $N-1$  vertices, where we have *contracted*  $\alpha$ , so that its two endpoints<sup>2</sup> become a single vertex, and  $\alpha$  is gone. Delete loops if the contraction creates such. If contraction creates a multi-graph, it is a matter of taste whether to delete extra edges.

**2: Deletion-contraction Thm.** *On  $N$ -vertex loopless multi-graph  $G = (\mathbb{V}, \mathbb{E})$ , function  $\mathcal{P}_G()$  is a degree- $N$  monic polynomial. For each  $\alpha \in \mathbb{E}$ ,*

$$2a: \quad \mathcal{P}_G(x) = \mathcal{P}_{G \setminus \{\alpha\}}(x) - \mathcal{P}_{G/\alpha}(x),$$

as polynomials. Also,  $\mathcal{P}_G(x)$  has no constant term, except when  $G$  is the void graph.  $\diamond$

**Pf.** The only  $N=0$  graph is void, and  $\mathcal{P}_{\text{Emp}_0}(x)$  is constant-1, which is monic and of degree zero. Fixing  $N \geq 1$ , we induct on the number,  $L$ , of edges. The  $L=0$  case is trivial, since  $\mathcal{P}_{\text{Emp}_N}(x)$  is  $x^N$ .

As  $G \setminus \{\alpha\}$  has  $L-1$  edges, poly  $\mathcal{P}_{G \setminus \{\alpha\}}$  is monic of degree- $N$ . As  $G/\alpha$  has at-most  $L-1$  (we might have deleted a loop) and  $N-1$  vertices, our  $\mathcal{P}_{G/\alpha}$  is a degree- $[N-1]$  poly. Since their difference is a monic degree- $N$  poly, ISTE Establish the (2a) recurrence. We will show that

$$2b: \quad \mathcal{P}_G(k) + \mathcal{P}_{G/\alpha}(k) = \mathcal{P}_{G \setminus \{\alpha\}}(k),$$

for each posint  $k$ . This implies equality as polynomials, since we will have equal outputs, for  $N+1$  many values of  $k$ .

The endpoints of  $\alpha$ , call them  $\mathbf{u}$  and  $\mathbf{v}$ . Consider a coloring of  $G/\alpha$ , but split apart the combined vertex back into separate vertices  $\mathbf{u}$  and  $\mathbf{v}$  [and don't put in edge  $\alpha$ ]. This is now a coloring of  $G \setminus \{\alpha\}$  that gives  $\mathbf{u}$  and  $\mathbf{v}$  the *same* color. In contrast, each coloring of  $G$  gives distinct colors to  $\mathbf{u}$  and  $\mathbf{v}$ ; so removing  $\alpha$  gives a coloring of  $G \setminus \{\alpha\}$  with  $\mathbf{u}$  and  $\mathbf{v}$  having *different* colors. Hence (2b).  $\diamond$

<sup>2</sup>If  $\mathbf{u}, \mathbf{v}$  have multiple edges, then contracting a  $\mathbf{u} - \mathbf{v}$  edge creates a loop, hence a graph with  $\mathcal{P}() \equiv 0$ . This is ok, but inefficient; typically, first collapse each multi-edge to a single edge.

**Properties of  $\mathcal{P}_G$ .** Initially,  $N \in \mathbb{N}$ .

$$3a: \quad \mathcal{P}_{\text{Emp}_N}(x) = x^N.$$

$$\mathcal{P}_{K_N}(x) = \llbracket x \downarrow N \rrbracket.$$

Now,  $N \geq 1$ . The  $N$ -vertex *path graph*,  $P_N$ , is a special case of a tree. Below,  $T_N$  is an arbitrary tree on  $N$  vertices. Easily,

$$3b: \quad \mathcal{P}_{P_N}(x) = \mathcal{P}_{T_N}(x) = x \cdot [x-1]^{N-1}.$$

For  $N \geq 2$ ,

$$3c: \quad \mathcal{P}_{C_N}(x) = [x-1]^N + [-1]^N[x-1].$$

In particular,

$$3c': \quad \begin{aligned} \mathcal{P}_{C_4}(x) &= x \cdot [x-1] \cdot [x^2 - 3x + 3] \\ &= x^4 - 4x^3 + 6x^2 - 3x. \end{aligned}$$

**Proof of (3c).** Note  $C_2$  becomes the path  $P_2$ , after collapsing the multi-edge, hence has chrom-poly  $x \cdot [x-1]$ , which is what (3c) <sub>$N=2$</sub>  equals; the base case.

Applying (2a) to  $G := C_{N+1}$  and an edge  $\alpha$ , gives  $G \setminus \{\alpha\} = P_{N+1}$  and  $G/\alpha = C_N$ . So  $\mathcal{P}_G(x)$  equals

$$[x \cdot [x-1]^N] - [[x-1]^N + [-1]^N[x-1]].$$

And this reduces to  $[x-1]^{N+1} + [-1]^{N+1}[x-1]$ .  $\diamond$

From each vertex of  $C_N$ , attach an edge to a common vertex,  $\mathbf{u}_{N+1}$ . This *wheel graph*  $W_{N+1}$ , has  $N+1$  vertices and  $2N$  edges. For  $N \geq 2$ , then,

$$3d: \quad \begin{aligned} \mathcal{P}_{W_{N+1}}(x) &= x \cdot \mathcal{P}_{C_N}(x-1) \\ &\stackrel{\text{note}}{=} x [[x-2]^N + [-1]^N[x-2]]. \end{aligned}$$

(Fixing a posint  $x$ , there are  $x$  choices to color vertex  $\mathbf{u}_{N+1}$ , hence  $x-1$  colors available for the embedded  $C_N$ .) E.g, (??') gives  $\mathcal{P}_{C_4}(x-1) = [x-1] \cdot [x-2] \cdot [x^2 - 5x + 7]$ . So

$$3d': \quad \mathcal{P}_{W_5}(x) = \llbracket x \downarrow 3 \rrbracket \cdot [x^2 - 5x + 7].$$

**Trivial graphs.** Note  $\mathcal{P}_{C_1} = x$  and  $\mathcal{P}_{C_0} = \mathcal{P}_{\text{Emp}_0} = 1$ ; neither produced by (3c). The wheel-recurrence  $\mathcal{P}_{W_N}(x) = x \cdot \mathcal{P}_{C_{N-1}}(x-1)$  holds  $\forall N \geq 1$ .  $\square$

**Exer E1:** Suppose  $\mathcal{P}_G(x) = x \cdot [x-1]^{N-1}$ . Prove that  $G$  is a tree.

**Cone over a graph.** Write this: Wheel graph  $W_{N+1}$  is the cone over  $C_N$ . Double cone. And  $K$ -vertex cone gives clumsy formula involving bell numbers.

Alternatively, the  $K$ -vertex cone over  $G$  is the full-product of  $G$  with  $\text{Emp}_K$ .

**Defn.** An *alternating polynomial*  $h(x)$  has form

$$4: \quad B_N x^N - B_{N-1} x^{N-1} + B_{N-2} x^{N-2} - B_{N-3} x^{N-3} + \dots + [-1]^j B_{N-j} x^{N-j} + \dots + [-1]^{N-K} B_K x^K,$$

where  $N \geq K$  are natnums, and each  $B_j > 0$ . Call index  $K$  “the *low-degree* of  $h$ ”, written  $\text{LD}(h)$ . Here is an easy exercise. For  $f$  and  $g$  alternating-polys:

4a: Product  $f \cdot g$  is alternating,  
and  $\text{LD}(f \cdot g) = \text{LD}(f) + \text{LD}(g)$ .

4b: If  $\text{Deg}(f) = 1 + \text{Deg}(g)$ , then  $f - g$  is alternating, with  $\text{LD}(f - g) = \text{Min}(\text{LD}(f), \text{LD}(g))$ .  $\square$

**4c: Chromatic polynomial Theorem.** For a non-void simple graph  $G = (\mathbb{V}, \mathbb{E})$ , write its chromatic polynomial  $\mathcal{P}_G(x)$  in form (4). Then

\*:  $\mathcal{P}_G$  is a monic alternating intpoly, with  
 $N = |\mathbb{V}|$ ,  $B_{N-1} = |\mathbb{E}|$  and  $K = \mathbf{c}(G)$ .  $\diamond$

[This  $\mathbf{c}(G)$  is the number of connected-components.]

**Proof.** [We have “monic” and “ $N = |\mathbb{V}|$ ” from (2).] First suppose  $G$  decomposes into (non-void) disjoint subgraphs  $H_1 \sqcup H_2$ . Let  $N_j := |\mathbb{V}_{H_j}|$ ,  $L_j := |\mathbb{E}_{H_j}|$ ,  $K_j := \mathbf{c}(H_j)$  and  $f_j := \mathcal{P}_{H_j}$ . So  $f_j$  has form

$$f_j(x) = x^{N_j} - L_j x^{N_j-1} + \dots + C_j x^{K_j}$$

with  $C_j \neq 0$ . Easily,  $\mathcal{P}_G = f_1 \cdot f_2$ , hence is alternating, by (4a), with low-degree  $K_1 + K_2 \stackrel{\text{note}}{=} \mathbf{c}(G)$ . The penultimate coeff of  $f_1 \cdot f_2$  is  $[-L_1 + L_2]$ , which indeed is the number of  $G$ -edges. So *WLOG*,  $G$  is connected.

**When  $G$  is connected.** [Recall  $N \geq 1$  since  $G$  is non-void.] Pick a  $G$ -edge,  $\alpha$ , whose removal does not disconnect  $G$ ; if there is none such, then  $G$  is a tree [possibly the edgeless tree], where (\*) evidently holds.

Hence both  $G \setminus \{\alpha\}$  and  $G/\alpha$  are connected. Thus  $f := \mathcal{P}_{G \setminus \{\alpha\}}$  and  $g := \mathcal{P}_{G/\alpha}$  each satisfy (\*). So  $f - g$  is alternating, by (4b), and  $\text{LD}(f - g) = \text{Min}(1, 1) = 1$ , which is indeed the number of connected-comps of  $G$ .

**Counting edges.** Let  $L := |\mathbb{E}|$ . Our  $G \setminus \{\alpha\}$  has  $L-1$  edges, thus  $f(x) = x^N - [L-1]x^{N-1} + \dots$ . And  $g$  is monic,  $g(x) = x^{N-1} - \dots$ . The difference thus has form  $\mathcal{P}_G(x) = x^N - Lx^{N-1} + \dots$ , as desired.  $\spadesuit$

**4d: Chromatic-polynomial Corollary.** Polynomial  $\mathcal{P}_G$  has no negative roots. Setting  $K := \chi(G)$ , we can therefore write  $\mathcal{P}_G(x)$  in chromatic form as

$$x^{e_0} \cdot [x-1]^{e_1} \cdot [x-2]^{e_2} \cdots [x-[K-1]]^{e_{K-1}} \cdot f(x),$$

with each  $e_j \in \mathbb{Z}_+$ . Moreover,  $f$  is [either absent, or] a monic intpoly, with no negative real  $\heartsuit^3$  roots, and no rational roots.  $\diamond$

**Proof.** An alternating-poly evaluated at a negative real, yields a sum of posreals, hence is positive. Finally, since  $f$  is *primitive* (the GCD of its coeffs is 1) each rational root must be integral, by the *Gauss Lemma* for polynomials.

Exer E3.1415: “The composition of two chromatic-polys is always a chromatic-poly.” Prove, or CEX.

## Gluing

The next result uses

5: ... two graphs  $H_j$ , for  $j = 1, 2$ , with  $N_j$  many vertices and  $L_j$  many edges. Let  $h_j() := \mathcal{P}_{H_j}()$ .

5a: **Gluing lemma.** When  $G$  is built from non-void simple graphs  $H_1$  and  $H_2$  by...

0: **disjoint union**, then  $\mathcal{P}_G(x) = h_1(x) \cdot h_2(x)$ .

1: picking a vertex  $\mathbf{u}_j$  in  $H_j$  and identifying the two vertices, then  $\mathcal{P}_G(x) = h_1(x) \cdot h_2(x) / x$ . This  $G$  has  $N_1 + N_2 - 1$  many vertices and  $L_1 + L_2$  many edges. Call this  $G$  a *point-gluing* of  $H_1$  and  $H_2$ .

2: picking an edge  $\alpha_j$  in  $H_j$  and identifying the two edges (choose an orientation), then

$$\mathcal{P}_G(x) = h_1(x) \cdot h_2(x) / x[x-1].$$

This  $G$  has  $N_1 + N_2 - 2$  vertices and  $L_1 + L_2 - 1$  edges. Call this  $G$  an *edge-gluing* of  $H_1$  with  $H_2$ .

$\heartsuit^3$  However,  $f$  can have complex roots with negative real-part.

3: picking a vertex  $\mathbf{u}_j$  in  $H_j$  and putting in a (new) edge between them, then

$$\mathcal{P}_G(x) = h_1(x) \cdot h_2(x) \cdot \frac{x-1}{x}. \quad \diamond$$

This  $G$  has  $N_1 + N_2$  many vertices and  $L_1 + L_2 + 1$  many edges. This  $G$  is a **new-edge-attaching** of  $H_1$  and  $H_2$ .

**Proof.** Exercise. ◆

5b: **Gluing on a subgraph.** Graph  $M = (\mathbb{V}, \mathbb{E})$  is a **subgraph** of  $H = (\mathbb{V}', \mathbb{E}')$  if there exist injections  $\Phi: \mathbb{V} \hookrightarrow \mathbb{V}'$  and  $\Psi: \mathbb{E} \hookrightarrow \mathbb{E}'$  so that:

For each  $\alpha \in \mathbb{E}$  with endpoints  $\mathbf{u}, \mathbf{v} \in \mathbb{V}$ , necessarily, the endpoints of  $\Psi(\alpha)$  are  $\Phi(\mathbf{u})$  and  $\Phi(\mathbf{v})$ .

Henceforth, for both injections we'll use a common symbol (usually  $\Phi$ ) and write  $\boxed{\Phi: M \hookrightarrow H}$ .

Consider graphs  $H_j$  as in (5), as well as a graph  $M$ . Suppose  $\Phi_1: M \hookrightarrow H_1$  and  $\Phi_2: M \hookrightarrow H_2$ . Define the

**gluing** of  $H_1$  with  $H_2$ , over  $(\Phi_1, \Phi_2)$

as the graph  $G$  which is the “union” of  $H_1$  and  $H_2$ , where for each vertex  $\mathbf{u}$  and edge  $\alpha$  of  $M$ :

5c: Vertex  $\Phi_1(\mathbf{u})$  is identified with  $\Phi_2(\mathbf{u})$  and edge  $\Phi_1(\alpha)$  is identified with  $\Phi_2(\alpha)$ .

So  $G$  has  $N_1 + N_2 - |\text{Vertices}(M)|$  many vertices, and  $L_1 + L_2 - |\text{Edges}(M)|$  many edges. When we don't need the details of the gluing, we will refer to  $G$  as

a gluing of  $H_1$  with  $H_2$ , over  $M$ .

**NOTATION:** I use symbol  $\sqcup$  for “**disjointunion**”, so let's use  $H_1 \sqcup H_2$  for gluing over the *void graph*. More generally, use

$$5d: H_1 \sqcup_M H_2 \quad \text{or} \quad H_1 \sqcup_{\Phi_2}^{\Phi_1} H_2$$

for gluing over  $M$ ; the latter, if the details are needed.

Say that  $M$  is **gluing-good** if, for all graphs  $H_1, H_2$  having  $M$  as a subgraph, necessarily

$$5e: \mathcal{P}_G(x) = \mathcal{P}_{H_1}(x) \cdot \mathcal{P}_{H_2}(x) / \mathcal{P}_M(x),$$

whenever  $G$  is a gluing of  $H_1$  with  $H_2$  over  $M$ . □

**Exer E2.1:** Find an infinite family of connected graphs which are gluing-good.

**Exer E2.2:** “Each connected graph  $M$  is gluing-good”. Find a proof, or CEX.

**Bipartite graphs.** For natural numbers  $B$  and  $G$ , the complete bipartite graph  $K_{B,G}$  has all edges between  $\mathbf{B} := [1..B]$  and  $\mathbf{G} := [1..G]$ , the “Boys” and “Girls”, and no other edges.

For natnums  $B$  and  $\ell$ , the “**Stirling number** of the second kind”,  $\mathcal{S}(B, \ell)$ , is the number of partitions of  $[1..B]$  into  $\ell$  many non-empty atoms. (I.e,  $\mathcal{S}(B, \ell)$  is the number of equivalence relations on  $[1..B]$  that have  $\ell$  many [non-void] equivalence classes.)

**6: Sara's Lemma.** Let  $f_{B,G}$  abbreviate  $\mathcal{P}_{K_{B,G}}$ . Then

$$6a: f_{B,G}(x) = \sum_{\ell=0}^B \mathcal{S}(B, \ell) \cdot [x \downarrow \ell] \cdot [x - \ell]^G. \quad \diamond$$

**Computing.** At  $B = 0$ , the RhS is  $1 \cdot 1 \cdot [x - 0]^G$  which is  $x^G$ , which is correct.

Once  $B \geq 1$  we can start the sum at  $\ell = 1$ , since there are no partitions of the empty set into positively many atoms.

**CASE:  $B = 1$ .** The RhS is  $1 \cdot [x \downarrow 1] \cdot [x - 1]^G$ , i.e,  $x \cdot [x - 1]^G$ , which is what (3b) says, as  $K_{1,G}$  is a tree.

**CASE:  $B = 2$ .** At  $B = 2$ , our RhS(6a) is

$$\begin{aligned} \mathcal{S}(2, 1) \cdot [x \downarrow 1] \cdot [x - 1]^G &= 1 \cdot x \cdot [x - 1]^G \\ + &+ \\ \mathcal{S}(2, 2) \cdot [x \downarrow 2] \cdot [x - 2]^G &= 1 \cdot x[x-1] \cdot [x - 2]^G. \end{aligned}$$

So  $f_{2,G}(x)$  equals  $x \cdot [x - 1]^G + [x-1] \cdot [x - 2]^G$ .

Once  $G \geq 1$ , we have that

$$6b: f_{2,G}(x) = x \cdot [x-1] \cdot [x - 1]^{G-1} + [x - 2]^G.$$

Plugging in  $G=2$  yields RhS(??'), which is reassuring seeing as  $K_{2,2}$  equals  $C_4$ .

**CASE:  $B = 3$ .** RhS(6a) is a sum of three terms:

$$\begin{aligned} \mathcal{S}(3, 1) \cdot [x \downarrow 1] \cdot [x - 1]^G &= 1 \cdot x[x - 1]^G; \\ \mathcal{S}(3, 2) \cdot [x \downarrow 2] \cdot [x - 2]^G &= 3 \cdot x[x - 1][x - 2]^G; \\ \mathcal{S}(3, 3) \cdot [x \downarrow 3] \cdot [x - 3]^G &= 1 \cdot x[x - 1][x - 2][x - 3]^G. \end{aligned}$$

As soon as  $G \geq 1$ , ratio  $f_{3,G}(x)/x[x - 1]$  equals

$$6c: [x - 1]^{G-1} + 3[x - 2]^G + [x - 2][x - 3]^G.$$

Using colors orange&blue there are 2 colorings of  $K_{3,G}$  [since  $G>0$ ], so (6c) at  $x=2$  better equal 1. Does it?

When  $G = 1$ , the  $K_{3,1}$  is a 4-vertex tree, so (6c) with  $G=1$  better darn well be  $[x-1]^2$ . Is it?

FWIW,  $(6c)_{G=2}$  is  $[x^3 - 5x^2 + 10x - 7]$ ; irred.

Evaluating,  $(6c)_{G=3}$  gives

$$6d: \quad \frac{f_{3,3}(x)}{x \cdot [x-1]} = x^4 - 8x^3 + 28x^2 - 47x + 31.$$

This last is irreducible over the rationals.  $\square$

## Graph-Stirling numbers

For a generalization of “bipartite graph”, consider two graphs  $H_j = (\mathbb{V}_j, \mathbb{E}_j)$  with  $N_j$  vertices and  $L_j$  edges. The **full product**  $G := H_1 \odot H_2$  is their disjoint union, augmented by an edge from each  $H_1$ -vertex, to each  $H_2$ -vertex. Thus

$$7a: \quad H_1 \odot H_2 \text{ has } N_1 + N_2 \text{ many vertices, and } L_1 + L_2 + [N_1 N_2] \text{ many edges.}$$

We'll get a formula for its chrom-poly, in terms of the following type of polynomial product.

Given two polynomials  $f$  and  $g$ , we define their **falling product**  $h := f \dashv g$ , as following.

❶ Write each w.r.t the *falling-factorial* basis, i.e

$$f = \sum_{j=0}^J \alpha_j \mathbf{t}_j \quad \text{and} \quad g = \sum_{k=0}^K \beta_k \mathbf{t}_k,$$

as shown in (1b).

❷ Compute  $\vec{\gamma} := \vec{\alpha} \circledast \vec{\beta} = (\gamma_0, \gamma_1, \dots, \gamma_{J+K})$ , the convolution.

❸ Define  $h := \sum_{n=0}^{J+K} \gamma_n \mathbf{t}_n$ .

**Defn.** A **Stirling partition** of graph  $G = (\mathbb{V}, \mathbb{E})$ , is a partition of  $\mathbb{V}$  into non-empty subsets (called the **atoms** of the ptn) so that no two adjacent vertices are in the same atom. [I.e, each atom is an “independent set”.]

For natnum  $\ell$ , define the **graph-Stirling number**

$$7b: \quad \mathcal{S}(G, \ell)$$

to be the number of  $\ell$ -atom Stirling partitions of  $\mathbb{V}$ .

**7c: Graph-Stirling Thm.** For an  $N$ -vertex<sup>24</sup> graph  $G$ ,

$$\mathcal{P}_G(x) = \sum_{\ell=0}^N \mathcal{S}(G, \ell) \cdot \llbracket x \downarrow \ell \rrbracket. \quad \diamond$$

**Proof.** Exercise.  $\diamond$

**7d: Full-product Thm.** Consider a graph  $G := H_1 \odot H_2$ . Then

$$\mathcal{P}_G = \mathcal{P}_{H_1} \dashv \mathcal{P}_{H_2}. \quad \diamond$$

**Proof idea.** Fix  $\ell$ . The  $\ell$ -atom Stirling partitions of  $G$  are in 1-to-1 correspondence with: Pick natnums with  $j_1 + j_2 = \ell$ , then take a  $j_1$ -atom Stirling ptn of  $H_1$ , and a  $j_2$ -atom Stirling ptn of  $H_2$ .  $\diamond$

**7e: Example.** Let  $H$  be  $P_2 \sqcup P_1$ . So  $\mathcal{S}(H, 1) = 0$ ,  $\mathcal{S}(H, 2) = 2$  and  $\mathcal{S}(H, 3) = 1$ . Our (7c) asserts that  $\mathcal{P}_H = \sum_{j=0}^3 \alpha_j \mathbf{t}_j$ , where  $\vec{\alpha} = (0, 0, 2, 1)$ . I.e,

$$\begin{aligned} \mathcal{P}_H(x) &= 2 \cdot \llbracket x \downarrow 2 \rrbracket + \llbracket x \downarrow 3 \rrbracket \\ &= \llbracket x \downarrow 2 \rrbracket [2 + [x - 2]] \stackrel{\text{note}}{=} x^2 \cdot [x - 1]. \end{aligned}$$

This agrees with (3b) and with (5a.0).

Let's compute  $G := H \odot K$ , where  $K := K_1$ . Our  $G$  is a  $K_3$  with a new edge attached to a vertex. So (5a.3) says  $\mathcal{P}_G(x) = \llbracket x \downarrow 3 \rrbracket \cdot [x - 1]$ . What does (7d) say?

Our  $K$  has chr-poly  $x = \llbracket x \downarrow 1 \rrbracket$ . We set  $\vec{\beta} := (0, 1)$ , then compute convolution  $\vec{\gamma} := \vec{\alpha} \circledast \vec{\beta} = (0, 0, 0, 2, 1)$ . Thm (7d) asserts that

$$\begin{aligned} \mathcal{P}_G(x) &= 2 \cdot \llbracket x \downarrow 3 \rrbracket + \llbracket x \downarrow 4 \rrbracket \\ &= \llbracket x \downarrow 3 \rrbracket [2 + [x - 3]] \stackrel{\text{note}}{=} \llbracket x \downarrow 3 \rrbracket [x - 1]. \end{aligned}$$

This agrees with our derivation via (5a.3).  $\square$

**7f: Bipartite ex.** Graph  $H := \text{Emp}_3$  has  $\mathcal{S}(H, 1) = 1$ ,  $\mathcal{S}(H, 2) = 3$ ,  $\mathcal{S}(H, 3) = 1$ , and so  $\vec{\alpha} = (0, 1, 3, 1)$ . Thus

$$\begin{aligned} \mathcal{P}_H(x) &= \llbracket x \downarrow 1 \rrbracket + 3 \llbracket x \downarrow 2 \rrbracket + \llbracket x \downarrow 3 \rrbracket \\ &= x \cdot [1 + [x - 1][3 + [x - 2]]] \\ &= x \cdot [1 + [x^2 - 1]] = x^3. \end{aligned}$$

<sup>24</sup>We can start the sum at  $\ell=1$  except when  $G$  is the void graph. After all, when  $N$  is positive then  $\mathcal{S}(G, 0)$  is zero.

This indeed agrees with (3a).

Since  $K_{3,3}$  is the full-product of  $H$  with  $H$ , we compute  $\vec{\alpha} := \vec{\alpha} \circledast \vec{\alpha} = (0, 0, 1, 6, 11, 6, 1)$ . Polynomial

$$[x \downarrow 2] + 6[x \downarrow 3] + 11[x \downarrow 4] + 6[x \downarrow 5] + [x \downarrow 6]$$

simplifies (thanks, *Maple*) to

$$x \cdot [x \downarrow 1] \cdot [x^4 - 8x^3 + 28x^2 - 47x + 31].$$

Happily, this agrees with (6d).  $\square$

### Spanning subgraphs

Fix a  $G = (\mathbb{V}, \mathbb{E})$  with  $N$  vertices and  $L$  edges. Each subset  $S \subset \mathbb{E}$  can be interpreted as  $(\mathbb{V}, S)$ , a spanning subgraph of  $G$ . [Thus  $G$  has  $2^L$  many spanning subgraphs.] Let  $\mathbf{c}(S)$  denote the number of connected components of  $S$ .

**8: CPSS Thm (Chromatic-Poly Spanning Subgraph).** *The chromatic polynomial of  $G$  satisfies*

$$*: \quad \mathcal{P}_G(x) = \sum_{S: S \subset \mathbb{E}} (-1)^{|S|} \cdot x^{\mathbf{c}(S)}. \quad \diamond$$

**Exer E4:** Prove this. Think *Inclusion-Exclusion*.  
[Hint: See the pamphlet on our TEACHING PAGE.]

**Comparison.** We can paraphrase Theorems (8) and (7c) as saying: *Spanning subgraphs* express  $\mathcal{P}_G$  w.r.t the standard basis  $(\mathbf{e}_j)_{j=0}^{\infty}$  [see (1b)], whereas *Graph-Stirling numbers* write  $\mathcal{P}_G$  w.r.t the falling-factorial basis  $(\mathbf{t}_n)_{n=0}^{\infty}$ .  $\square$

**8a: Ex.** As  $K_3$  has 3 edges, it has  $2^3 = 8$  spanning subgraphs. The no-edge graph has 3 CCs [Connected Components], hence contributes a  $[-1]^0 \cdot x^3 = x^3$  term.

The three one-edge spanning subgraphs each have 2 CCs; these contribute  $3 \cdot [-1]^1 \cdot x^2 = -3x^2$ .

The three two-edge spanning subgraphs, each have 1 CC, contributing  $3 \cdot [-1]^2 \cdot x^1 = 3x$ . Finally, the unique three-edge spanning subgraph has 1 CC, hence contributing a  $[-1]^3 \cdot x^1 = -x$  term. Adding,

$$\begin{aligned} \mathcal{P}_{K_3}(x) &= x^3 - 3x^2 + 3x - x \\ &= x^3 - 3x^2 + 2x \stackrel{\text{note}}{=} [x \downarrow 3]. \end{aligned} \quad \square$$

### Applications/Extensions

Here is one:

**9a: Orientations.** An *orientation* of  $G = (\mathbb{V}, \mathbb{E})$  is putting a direction on each edge, creating a digraph; so  $G$  has  $2^{|\mathbb{E}|}$  many orientations. An orientation of  $G$  is *acyclic* if it has no directed-cycles. Use  $\mathcal{A}(G)$  for the number of acyclic orientations of  $G$ .

An orientation of  $K_N$  is called an *N-tournament*; there are  $2^{\binom{N}{2}}$  of them. E.g.,  $K_4$  has  $2^6 = 64$  orientations. **Exer E5:** Prove that  $\mathcal{A}(K_N)$  equals  $[N]!$ .  $\square$

**9b: Acyclic-count Thm (Richard Stanley).** *On  $N$ -vertex  $G$ ,*

$$\mathcal{A}(G) = [-1]^N \cdot \mathcal{P}_G(-1). \quad \diamond$$

**Sketch.** The idea is to establish an analog of (2a),

$$9c: \quad \mathcal{A}(G) = \mathcal{A}(G \setminus \{\alpha\}) + \mathcal{A}(G/\alpha),$$

for each edge  $\alpha$  of  $G$ , as follows. Let  $M := G \setminus \{\alpha\}$ .

- Each acyclic-orient of  $M$  extends to *at least one* acyclic-orient of  $G$ .
- The number of acyclic-orient of  $M$  which give rise to *two* acyclic-orient of  $G$ , is  $\mathcal{A}(G/\alpha)$ .  $\diamond$

**10: Generalizing full-product.** Let  $\mathcal{G}$  be the set of all graphs. Consider a graph  $S = (\mathbb{V}, \mathbb{E})$  and a fnc  $\mathcal{F}: \mathbb{V} \rightarrow \mathcal{G}$ ; we'll often write  $\mathcal{F}(\mathbf{u})$  as  $\mathcal{F}_{\mathbf{u}}$ . Use  $N_{\mathbf{u}}|L_{\mathbf{u}}$  for the number of vertices|edges of  $\mathcal{F}_{\mathbf{u}}$ .

The “*full product* of  $\mathcal{F}$  over  $S$ ”

$$G := \bigodot_S(\mathcal{F})$$

is  $S$ , but where each  $S$ -vertex  $\mathbf{u}$  has been replaced by a copy of graph  $\mathcal{F}_{\mathbf{u}}$ . Moreover, for each  $S$ -edge  $\mathbf{v} \rightarrow \mathbf{w}$ :

Each  $\mathcal{F}_{\mathbf{v}}$ -vertex  $\mathbf{v}'$  and each  $\mathcal{F}_{\mathbf{w}}$ -vertex  $\mathbf{w}'$ , are the endpoints of a  $G$ -edge.

Thus  $\bigodot_S(\mathcal{F})$  has  $\sum_{\mathbf{u} \in \mathbb{V}} N_{\mathbf{u}}$  many vertices. It has

$$10a: \quad \left[ \sum_{\mathbf{u} \in \mathbb{V}} L_{\mathbf{u}} \right] + \sum_{\mathbf{v}, \mathbf{w}} [N_{\mathbf{v}} \cdot N_{\mathbf{w}}]$$

many edges, where the second sum is over all  $S$ -edges  $\mathbf{v} \rightarrow \mathbf{w}$ . In the case that  $S$  is a single edge,  $P_2$ , we recover the full-product as defined above (7a).

Full-product from (7a) is associative, so expressions such as  $H_1 \odot H_2 \odot \dots \odot H_5$  make sense. If we define  $\mathcal{F}(j) := H_j$ , then

$$10b: \quad H_1 \odot H_2 \odot \dots \odot H_5 = \bigodot_{K_5}(\mathcal{F}),$$

regarding  $[1..5]$  as the vertex-set of  $K_5$ . \square

Exer: Is there a formula for the chromatic number/polynomial of  $\bigodot_S(\mathcal{F})$ , in terms of corresponding information about  $S$  and function  $\mathcal{F}$ ? What about special cases, e.g.  $S = P_3$ ? Or  $S = C_{\text{Even}}$ ?

Filename: [Problems/GraphTheory/chromatic-num.latex](#)  
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