

Chromatic polynomial of a graph

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Polynomial preliminaries. A poly(nomial) such as $f(x) = 8x^2 - 7x + 2$ can be written as $8\mathbf{e}_2 - 7\mathbf{e}_1 + 2\mathbf{e}_0$, where $\mathbf{e}_j(x) := x^j$. As a linear combination,

$$\dagger: \quad f = \sum_{j=0}^2 \alpha_j \mathbf{e}_j,$$

where $\vec{\alpha} := (\alpha_0, \alpha_1, \alpha_2) = (2, -7, 8)$. The product of f with a degree-3 polynomial $g = \sum_{k=0}^3 \beta_k \mathbf{e}_k$ is

$$f \cdot g = \sum_{n=0}^5 \gamma_n \mathbf{e}_n,$$

where, summing over natnum-pairs (j, k) , each

$$1a: \quad \gamma_n = \sum_{(j,k): j+k=n} [\alpha_j \cdot \beta_k].$$

This $\vec{\gamma} = (\gamma_0, \dots, \gamma_5)$ is $\vec{\alpha} \circledast \vec{\beta}$, the *convolution* of $\vec{\alpha}$ with $\vec{\beta}$. An alternative notation to (\dagger) is

$$\ddagger: \quad f = \sum_{j=0}^{\infty} \alpha_j \mathbf{e}_j,$$

where $\alpha_j = 0$ when $j > 2$. Such an $\vec{\alpha} = (\alpha_0, \alpha_1, \dots)$ is an *eventually-zero* sequence.

We'll also need the *falling-factorial* polynomials,

$$[x \downarrow N] := x \cdot [x-1] \cdot [x-2] \cdots [x-[N-1]]$$

(a product of N many terms), where $N \in \mathbb{N}$. How to write a poly such as $F(x) := 2x^3 - x^2 - 3x + 3$ in terms of falling factorials? The coeff of x^3 is 2, so we subtract,

$$F(x) - 2 \cdot [x \downarrow 3] = 5x^2 - 7x + 3.$$

From this we subtract $5 \cdot [x \downarrow 2]$, producing

$$-2x + 3.$$

From *this* we subtract $-2[x \downarrow 1]$, yielding 3. From 3 we subtract $3 \cdot [x \downarrow 0]$, ending with zero. Thus

$$F(x) = 3 \cdot [x \downarrow 0] + -2 \cdot [x \downarrow 1] + 5 \cdot [x \downarrow 2] + 2 \cdot [x \downarrow 3].$$

Letting $\mathbf{t}_n(x) := [x \downarrow n]$, we have equality

$$1b: \quad \begin{aligned} F &= 3\mathbf{e}_0 + -3\mathbf{e}_1 + -1\mathbf{e}_2 + 2\mathbf{e}_3 \\ &= 3\mathbf{t}_0 + -2\mathbf{t}_1 + 5\mathbf{t}_2 + 2\mathbf{t}_3. \end{aligned}$$

We have done a change-of-basis computation, from basis $(\mathbf{e}_j)_{j=0}^{\infty}$ to the $(\mathbf{t}_n)_{n=0}^{\infty}$ basis. \square

Chromatic form. Looking ahead, consider a monic intpoly $\mathcal{P}()$ with $R \in \mathbb{N}$ many [not necessarily distinct] integer-roots Z_1, Z_2, \dots, Z_R . Writing $\mathcal{P}(x)$ in *chromatic form* means writing

$$1c: \quad \mathcal{P}(x) = \left[\prod_{j=1}^R [x - Z_j] \right] \cdot f(x),$$

where [either “ $\cdot f(x)$ ” is absent or] f is a monic intpoly with no integral roots. Indeed, courtesy the Gauss Lemma for polynomials, our f has *no rational roots*.

When \mathcal{P} is a chromatic polynomial, its integer-root part $\prod_j [x - Z_j]$ should be written in form

$$*: \quad x^{e_0} \cdot [x-1]^{e_1} \cdot [x-2]^{e_2} \cdots [x-[K-1]]^{e_{K-1}},$$

often mixed with falling-factorials, e.g

$$**: \quad [x \downarrow 2]^5 \cdot [x \downarrow 4] \cdot [x-1]^7.$$

See (4d), the Chromatic-polynomial Corollary. \square

Graph terminology. Use K_n for the *complete graph* on n vertices, and $K_{j,k}$ for the *complete (j, k) -bipartite graph*. Use C_n for the *cyclic graph* with n vertices and n edges [C_1 has a single self-edge; C_2 has 2 edges between 2 vertices; for $n \geq 3$, our C_n is a simple graph]. Use Emp_n for the n -vertex graph with *no* edges; an “Empty graph”.

Coloring. Consider $G = (\mathbb{V}, \mathbb{E})$, a finite graph, [loops, multiple-edges ok] with $N := |\mathbb{V}|$, and $L := |\mathbb{E}|$. Use $\mathbf{c}(G)$ for the # of *connected components* of G .

For $k \in \mathbb{N}$, a “*k-coloring* of G ” means to assign a “color”, an element of $[1..k]$, to each vertex, so that no two *neighbors* [the end-vertices of an edge] have the same color. A *k-coloring* is *full* if it uses all k colors. The *chromatic number* of G , written $\chi(G)$, is the minimum number of colors needed. I.e, it is the unique k so that there *is* a *k-coloring* of G , and every *k-coloring* of G is full.

For $k = 0, 1, 2, \dots$, let $\mathcal{P}_G(k)$ be the *number* of *k-colorings* of G . Note $\mathcal{P}_{\text{Emp}_0}(k) = 1$ since the *void graph* [no vertices] has exactly one *k-coloring*. Chromatic number $\chi(G)$ is the smallest natnum k for which $\mathcal{P}_G(k)$ is positive. \heartsuit

\heartsuit If G has a loop, then $\mathcal{P}_G() \equiv 0$ and $\chi(G) := \infty$.

If G has multiple edges between vertices \mathbf{u}, \mathbf{v} , then replacing them by a single edge will not change the chromatic poly/number. Without announcement we will do this; effectively, we compute chromatic polys only of *simple* graphs.

Deletion-contraction. Consider an N -vertex G and edge $\alpha \in \mathbb{E}$. Let $G \setminus \{\alpha\}$ mean to *delete* edge α ; no vertices are removed, so $G \setminus \{\alpha\}$ still has N vertices. In contrast, use G/α to mean the graph with $N-1$ vertices, where we have *contracted* α , so that its two endpoints^{♡2} become a single vertex, and α is gone. Delete loops if the contraction creates such. If contraction creates a multi-graph, it is a matter of taste whether to delete extra edges.

2: Deletion-contraction Thm. *On N -vertex loopless multi-graph $G = (\mathbb{V}, \mathbb{E})$, function $\mathcal{P}_G()$ is a degree- N monic polynomial. For each $\alpha \in \mathbb{E}$,*

$$2a: \quad \mathcal{P}_G(x) = \mathcal{P}_{G \setminus \{\alpha\}}(x) - \mathcal{P}_{G/\alpha}(x),$$

as polynomials. Also, $\mathcal{P}_G(x)$ has no constant term, except when G is the void graph. ♦

Pf. The only $N=0$ graph is void, and $\mathcal{P}_{\text{Emp}_0}(x)$ is constant-1, which is monic and of degree zero. Fixing $N \geq 1$, we induct on the number, L , of edges. The $L=0$ case is trivial, since $\mathcal{P}_{\text{Emp}_N}(x)$ is x^N .

As $G \setminus \{\alpha\}$ has $L-1$ edges, poly $\mathcal{P}_{G \setminus \{\alpha\}}$ is monic of degree- N . As G/α has at-most $L-1$ (we might have deleted a loop) and $N-1$ vertices, our $\mathcal{P}_{G/\alpha}$ is a degree- $[N-1]$ poly. Since their difference is a monic degree- N poly, ISTE establish the (2a) recurrence. We will show that

$$2b: \quad \mathcal{P}_G(k) + \mathcal{P}_{G/\alpha}(k) = \mathcal{P}_{G \setminus \{\alpha\}}(k),$$

for each posint k . This implies equality as polynomials, since we will have equal outputs, for $N+1$ many values of k .

The endpoints of α , call them \mathbf{u} and \mathbf{v} . Consider a coloring of G/α , but split apart the combined vertex back into separate vertices \mathbf{u} and \mathbf{v} [and don't put in edge α]. This is now a coloring of $G \setminus \{\alpha\}$ that gives \mathbf{u} and \mathbf{v} the *same* color. In contrast, each coloring of G gives distinct colors to \mathbf{u} and \mathbf{v} ; so removing α gives a coloring of $G \setminus \{\alpha\}$ with \mathbf{u} and \mathbf{v} having *different* colors. Hence (2b). ♦

^{♡2}If \mathbf{u}, \mathbf{v} have multiple edges, then contracting a $\mathbf{u} \text{---} \mathbf{v}$ edge creates a loop, hence a graph with $\mathcal{P}() \equiv 0$. This is ok, but inefficient; typically, first collapse each multi-edge to a single edge.

Properties of \mathcal{P}_G . Initially, $N \in \mathbb{N}$.

$$3a: \quad \begin{aligned} \mathcal{P}_{\text{Emp}_N}(x) &= x^N. \\ \mathcal{P}_{K_N}(x) &= \llbracket x \downarrow N \rrbracket. \end{aligned}$$

Now, $N \geq 1$. The N -vertex **path graph**, P_N , is a special case of a tree. Below, T_N is an arbitrary tree on N vertices. Easily,

$$3b: \quad \mathcal{P}_{P_N}(x) = \mathcal{P}_{T_N}(x) = x \cdot [x-1]^{N-1}.$$

For $N \geq 2$,

$$3c: \quad \mathcal{P}_{C_N}(x) = [x-1]^N + [-1]^N [x-1].$$

In particular,

$$3c': \quad \begin{aligned} \mathcal{P}_{C_4}(x) &= x \cdot [x-1] \cdot [x^2 - 3x + 3] \\ &= x^4 - 4x^3 + 6x^2 - 3x. \end{aligned}$$

Proof of (3c). Note C_2 becomes the path P_2 , after collapsing the multi-edge, hence has chrom-poly $x \cdot [x-1]$, which is what $(3c)_{N=2}$ equals; the base case.

Applying (2a) to $G := C_{N+1}$ and an edge α , gives $G \setminus \{\alpha\} = P_{N+1}$ and $G/\alpha = C_N$. So $\mathcal{P}_G(x)$ equals

$$[x \cdot [x-1]^N] - [[x-1]^N + [-1]^N [x-1]].$$

And this reduces to $[x-1]^{N+1} + [-1]^{N+1} [x-1]$. ♦

From each vertex of C_N , attach an edge to a common vertex, \mathbf{u}_{N+1} . This **wheel graph** W_{N+1} , has $N+1$ vertices and $2N$ edges. For $N \geq 2$, then,

$$3d: \quad \begin{aligned} \mathcal{P}_{W_{N+1}}(x) &= x \cdot \mathcal{P}_{C_N}(x-1) \\ &\stackrel{\text{note}}{=} x [[x-2]^N + [-1]^N [x-2]]. \end{aligned}$$

(Fixing a posint x , there are x choices to color vertex \mathbf{u}_{N+1} , hence $x-1$ colors available for the embedded C_N .) E.g., ($??'$) gives $\mathcal{P}_{C_4}(x-1) = [x-1] \cdot [x-2] \cdot [x^2 - 5x + 7]$. So

$$3d': \quad \mathcal{P}_{W_5}(x) = \llbracket x \downarrow 3 \rrbracket \cdot [x^2 - 5x + 7].$$

Trivial graphs. Note $\mathcal{P}_{C_1} = x$ and $\mathcal{P}_{C_0} = \mathcal{P}_{\text{Emp}_0} = 1$; neither produced by (3c). The wheel-recurrence $\mathcal{P}_{W_N}(x) = x \cdot \mathcal{P}_{C_{N-1}}(x-1)$ holds $\forall N \geq 1$. □

Exer E1: Suppose $\mathcal{P}_G(x) = x \cdot [x-1]^{N-1}$. Prove that G is a tree.

Cone over a graph. Write this: Wheel graph W_{N+1} is the cone over C_N . Double cone. And K -vertex cone gives clumsy formula involving bell numbers.

Alternatively, the K -vertex cone over G is the full-product of G with Emp_K .

Defn. An **alternating polynomial** $h(x)$ has form

$$4: \quad B_N x^N - B_{N-1} x^{N-1} + B_{N-2} x^{N-2} - B_{N-3} x^{N-3} + \dots + [-1]^j B_{N-j} x^{N-j} + \dots + [-1]^{N-K} B_K x^K,$$

where $N \geq K$ are natnums, and each $B_j > 0$. Call index K “the **low-degree** of h ”, written $\text{LD}(h)$. Here is an easy exercise. For f and g alternating-polys:

- 4a: Product $f \cdot g$ is alternating,
and $\text{LD}(f \cdot g) = \text{LD}(f) + \text{LD}(g)$.
- 4b: If $\text{Deg}(f) = 1 + \text{Deg}(g)$, then $f - g$ is alternating, with $\text{LD}(f - g) = \min(\text{LD}(f), \text{LD}(g))$. \square

4c: Chromatic polynomial Theorem. For a non-void simple graph $G = (\mathbb{V}, \mathbb{E})$, write its chromatic polynomial $\mathcal{P}_G(x)$ in form (4). Then

- *: \mathcal{P}_G is a monic alternating intpoly, with
 $N = |\mathbb{V}|$, $B_{N-1} = |\mathbb{E}|$ and $K = \mathbf{c}(G)$. \diamond

[This $\mathbf{c}(G)$ is the number of connected-components.]

Proof. [We have “monic” and “ $N = |\mathbb{V}|$ ” from (2).] First suppose G decomposes into (non-void) disjoint subgraphs $H_1 \sqcup H_2$. Let $N_j := |\mathbb{V}_{H_j}|$, $L_j := |\mathbb{E}_{H_j}|$, $K_j := \mathbf{c}(H_j)$ and $f_j := \mathcal{P}_{H_j}$. So f_j has form

$$f_j(x) = x^{N_j} - L_j x^{N_j-1} + \dots + C_j x^{K_j}$$

with $C_j \neq 0$. Easily, $\mathcal{P}_G = f_1 \cdot f_2$, hence is alternating, by (4a), with low-degree $K_1 + K_2 \stackrel{\text{note}}{=} \mathbf{c}(G)$. The penultimate coeff of $f_1 \cdot f_2$ is $-[L_1 + L_2]$, which indeed is the number of G -edges. So *WLOG*, G is connected.

When G is connected. [Recall $N \geq 1$ since G is non-void.] Pick a G -edge, α , whose removal does not disconnect G ; if there is none such, then G is a tree [possibly the edgeless tree], where (*) evidently holds.

Hence both $G \setminus \{\alpha\}$ and G/α are connected. Thus $f := \mathcal{P}_{G \setminus \{\alpha\}}$ and $g := \mathcal{P}_{G/\alpha}$ each satisfy (*). So $f - g$ is alternating, by (4b), and $\text{LD}(f - g) = \min(1, 1) = 1$, which is indeed the number of connected-comps of G .

Counting edges. Let $L := |\mathbb{E}|$. Our $G \setminus \{\alpha\}$ has $L - 1$ edges, thus $f(x) = x^N - [L - 1]x^{N-1} + \dots$. And g is monic, $g(x) = x^{N-1} - \dots$. The difference thus has form $\mathcal{P}_G(x) = x^N - Lx^{N-1} + \dots$, as desired. \blacklozenge

4d: Chromatic-polynomial Corollary. Polynomial \mathcal{P}_G has no negative roots. Setting $K := \chi(G)$, we can therefore write $\mathcal{P}_G(x)$ in chromatic form as

$$x^{e_0} \cdot [x - 1]^{e_1} \cdot [x - 2]^{e_2} \dots [x - [K - 1]]^{e_{K-1}} \cdot f(x),$$

with each $e_j \in \mathbb{Z}_+$. Moreover, f is [either absent, or] a monic intpoly, with no negative real^{♥3} roots, and no rational roots. \diamond

Proof. An alternating-poly evaluated at a negative real, yields a sum of posreals, hence is positive. Finally, since f is **primitive** (the GCD of its coeffs is 1) each rational root must be integral, by the Gauss Lemma for polynomials.

Exer E3.1415: “The composition of two chromatic-polys is always a chromatic-poly.” Prove, or CEX.

Gluing

The next result uses

- ...two graphs H_j , for $j = 1, 2$, with N_j
5: many vertices and L_j many edges. Let
 $h_j() := \mathcal{P}_{H_j}()$.

5a: Gluing lemma. When G is built from non-void simple graphs H_1 and H_2 by...

0: **disjoint union**, then $\mathcal{P}_G(x) = h_1(x) \cdot h_2(x)$.

1: **picking a vertex \mathbf{u}_j in H_j and identifying the two vertices**, then $\mathcal{P}_G(x) = h_1(x) \cdot h_2(x) / x$. This G has $N_1 + N_2 - 1$ many vertices and $L_1 + L_2$ many edges. Call this G a **point-gluing** of H_1 and H_2 .

2: **picking an edge α_j in H_j and identifying the two edges (choose an orientation)**, then

$$\mathcal{P}_G(x) = h_1(x) \cdot h_2(x) / x[x - 1].$$

This G has $N_1 + N_2 - 2$ vertices and $L_1 + L_2 - 1$ edges. Call this G an **edge-gluing** of H_1 with H_2 .

^{♥3}However, f can have complex roots with negative real-part.

3: picking a vertex \mathbf{u}_j in H_j and putting in a (new) edge between them, then

$$\mathcal{P}_G(x) = h_1(x) \cdot h_2(x) \cdot \frac{x-1}{x}. \quad \diamond$$

This G has $N_1 + N_2$ many vertices and $L_1 + L_2 + 1$ many edges. This G is a **new-edge-attaching** of H_1 and H_2 .

Proof. Exercise. ♦

5b: **Gluing on a subgraph.** Graph $M = (\mathbb{V}, \mathbb{E})$ is a **subgraph** of $H = (\mathbb{V}', \mathbb{E}')$ if there exist injections $\Phi: \mathbb{V} \hookrightarrow \mathbb{V}'$ and $\Psi: \mathbb{E} \hookrightarrow \mathbb{E}'$ so that:

For each $\alpha \in \mathbb{E}$ with endpoints $\mathbf{u}, \mathbf{v} \in \mathbb{V}$, necessarily, the endpoints of $\Psi(\alpha)$ are $\Phi(\mathbf{u})$ and $\Phi(\mathbf{v})$.

Henceforth, for both injections we'll use a common symbol (usually Φ) and write $(\Phi: M \hookrightarrow H)$.

Consider graphs H_j as in (5), as well as a graph M . Suppose $\Phi_1: M \hookrightarrow H_1$ and $\Phi_2: M \hookrightarrow H_2$. Define the

gluing of H_1 with H_2 , over (Φ_1, Φ_2)

as the graph G which is the “union” of H_1 and H_2 , where for each vertex \mathbf{u} and edge α of M :

5c: Vertex $\Phi_1(\mathbf{u})$ is identified with $\Phi_2(\mathbf{u})$ and edge $\Phi_1(\alpha)$ is identified with $\Phi_2(\alpha)$.

So G has $N_1 + N_2 - |\text{Vertices}(M)|$ many vertices, and $L_1 + L_2 - |\text{Edges}(M)|$ many edges. When we don't need the details of the gluing, we will refer to G as

a gluing of H_1 with H_2 , over M .

NOTATION: I use symbol \sqcup for “**disjointunion**”, so let's use $H_1 \sqcup H_2$ for gluing over the *void graph*. More generally, use

$$5d: \quad H_1 \sqcup_M H_2 \quad \text{or} \quad H_1 \sqcup_{\Phi_1}^{\Phi_2} H_2$$

for gluing over M ; the latter, if the details are needed.

Say that M is **gluing-good** if, for all graphs H_1, H_2 having M as a subgraph, necessarily

$$5e: \quad \mathcal{P}_G(x) = \mathcal{P}_{H_1}(x) \cdot \mathcal{P}_{H_2}(x) / \mathcal{P}_M(x),$$

whenever G is a gluing of H_1 with H_2 over M . □

Exer E2.1: Find an infinite family of connected graphs which are gluing-good.

Exer E2.2: “Each connected graph M is gluing-good”. Find a proof, or CEX.

Bipartite graphs. For natural numbers B and G , the complete bipartite graph $K_{B,G}$ has all edges between $\mathbf{B} := [1..B]$ and $\mathbf{G} := [1..G]$, the “Boys” and “Girls”, and no other edges.

For natnums B and ℓ , the “**Stirling number of the second kind**”, $\mathcal{S}(B, \ell)$, is the number of partitions of $[1..B]$ into ℓ many non-empty atoms. (I.e, $\mathcal{S}(B, \ell)$ is the number of equivalence relations on $[1..B]$ that have ℓ many [non-void] equivalence classes.)

6: **Sara's Lemma.** Let $f_{B,G}$ abbreviate $\mathcal{P}_{K_{B,G}}$. Then

$$6a: \quad f_{B,G}(x) = \sum_{\ell=0}^B \mathcal{S}(B, \ell) \cdot \llbracket x \downarrow \ell \rrbracket \cdot [x - \ell]^G. \quad \diamond$$

Computing. At $B = 0$, the RhS is $1 \cdot 1 \cdot [x - 0]^G$ which is x^G , which is correct.

Once $B \geq 1$ we can start the sum at $\ell = 1$, since there are no partitions of the empty set into positively many atoms.

CASE: $B = 1$. The RhS is $1 \cdot \llbracket x \downarrow 1 \rrbracket \cdot [x - 1]^G$, i.e, $x \cdot [x - 1]^G$, which is what (3b) says, as $K_{1,G}$ is a tree.

CASE: $B = 2$. At $B = 2$, our RhS(6a) is

$$\begin{array}{ccc} \mathcal{S}(2, 1) \cdot \llbracket x \downarrow 1 \rrbracket \cdot [x - 1]^G & = & 1 \cdot x \cdot [x - 1]^G \\ + & & + \\ \mathcal{S}(2, 2) \cdot \llbracket x \downarrow 2 \rrbracket \cdot [x - 2]^G & = & 1 \cdot x[x - 1] \cdot [x - 2]^G. \end{array}$$

So $f_{2,G}(x)$ equals $x \cdot [x - 1]^G + [x - 1] \cdot [x - 2]^G$.

Once $G \geq 1$, we have that

$$6b: \quad f_{2,G}(x) = x \cdot [x - 1] \cdot \left[[x - 1]^{G-1} + [x - 2]^G \right].$$

Plugging in $G=2$ yields RhS(??'), which is reassuring seeing as $K_{2,2}$ equals C_4 .

CASE: $B = 3$. RhS(6a) is a sum of three terms:

$$\begin{array}{l} \mathcal{S}(3, 1) \cdot \llbracket x \downarrow 1 \rrbracket \cdot [x - 1]^G = 1 \cdot x[x - 1]^G; \\ \mathcal{S}(3, 2) \cdot \llbracket x \downarrow 2 \rrbracket \cdot [x - 2]^G = 3 \cdot x[x - 1][x - 2]^G; \\ \mathcal{S}(3, 3) \cdot \llbracket x \downarrow 3 \rrbracket \cdot [x - 3]^G = 1 \cdot x[x - 1][x - 2][x - 3]^G. \end{array}$$

As soon as $G \geq 1$, ratio $f_{3,G}(x)/x[x - 1]$ equals

$$6c: \quad [x - 1]^{G-1} + 3[x - 2]^G + [x - 2][x - 3]^G.$$

Using colors *orange&blue* there are 2 colorings of $K_{3,G}$ [since $G > 0$], so (6c) at $x=2$ *better* equal 1. Does it?

When $G = 1$, the $K_{3,1}$ is a 4-vertex tree, so (6c) with $G=1$ *better darn well* be $[x-1]^2$. Is it?

FWIW, $(6c)_{G=2}$ is $[x^3 - 5x^2 + 10x - 7]$; irred.

Evaluating, $(6c)_{G=3}$ gives

$$6d: \quad \frac{f_{3,3}(x)}{x \cdot [x-1]} = x^4 - 8x^3 + 28x^2 - 47x + 31.$$

This last is irreducible over the rationals. \square

Graph-Stirling numbers

For a generalization of “bipartite graph”, consider two graphs $H_j = (\mathbb{V}_j, \mathbb{E}_j)$ with N_j vertices and L_j edges. The **full product** $G := H_1 \odot H_2$ is their disjoint union, augmented by an edge from each H_1 -vertex, to each H_2 -vertex. Thus

$$7a: \quad \begin{array}{l} H_1 \odot H_2 \text{ has } N_1 + N_2 \text{ many vertices, and} \\ L_1 + L_2 + [N_1 N_2] \text{ many edges.} \end{array}$$

We’ll get a formula for its chrom-poly, in terms of the following type of polynomial product.

Given two polynomials f and g , we define their **falling product** $h := f \downarrow g$, as following.

- ❶ Write each w.r.t the *falling-factorial* basis, i.e

$$f = \sum_{j=0}^J \alpha_j \mathbf{t}_j \quad \text{and} \quad g = \sum_{k=0}^K \beta_k \mathbf{t}_k,$$

as shown in (1b).

- ❷ Compute $\vec{\gamma} := \vec{\alpha} \circledast \vec{\beta} = (\gamma_0, \gamma_1, \dots, \gamma_{J+K})$, the convolution.

- ❸ Define $h := \sum_{n=0}^{J+K} \gamma_n \mathbf{t}_n$.

Defn. A **Stirling partition** of graph $G=(\mathbb{V}, \mathbb{E})$, is a partition of \mathbb{V} into non-empty subsets (called the **atoms** of the ptn) so that no two adjacent vertices are in the same atom. [I.e, each atom is an “independent set”.]

For natnum ℓ , define the **graph-Stirling number**

$$7b: \quad \mathcal{S}(G, \ell)$$

to be the number of ℓ -atom Stirling partitions of \mathbb{V} .

7c: Graph-Stirling Thm. For an N -vertex^{♥4} graph G ,

$$\mathcal{P}_G(x) = \sum_{\ell=0}^N \mathcal{S}(G, \ell) \cdot \llbracket x \downarrow \ell \rrbracket. \quad \diamond$$

Proof. Exercise. \diamond

7d: Full-product Thm. Consider a graph $G := H_1 \odot H_2$. Then

$$\mathcal{P}_G = \mathcal{P}_{H_1} \downarrow \mathcal{P}_{H_2}. \quad \diamond$$

Proof idea. Fix ℓ . The ℓ -atom Stirling partitions of G are in 1-to-1 correspondence with: *Pick natnums with $j_1 + j_2 = \ell$, then take a j_1 -atom Stirling ptn of H_1 , and a j_2 -atom Stirling ptn of H_2 .* \diamond

7e: Example. Let H be $P_2 \sqcup P_1$. So $\mathcal{S}(H, 1) = 0$, $\mathcal{S}(H, 2) = 2$ and $\mathcal{S}(H, 3) = 1$. Our (7c) asserts that $\mathcal{P}_H = \sum_{j=0}^3 \alpha_j \mathbf{t}_j$, where $\vec{\alpha} = (0, 0, 2, 1)$. I.e,

$$\begin{aligned} \mathcal{P}_H(x) &= 2 \cdot \llbracket x \downarrow 2 \rrbracket + \llbracket x \downarrow 3 \rrbracket \\ &= \llbracket x \downarrow 2 \rrbracket [2 + [x - 2]] \stackrel{\text{note}}{=} x^2 \cdot [x - 1]. \end{aligned}$$

This agrees with (3b) and with (5a.0).

Let’s compute $G := H \odot K$, where $K := K_1$. Our G is a K_3 with a new edge *attached* to a vertex. So (5a.3) says $\mathcal{P}_G(x) = \llbracket x \downarrow 3 \rrbracket \cdot [x - 1]$. What does (7d) say?

Our K has chr-poly $x = \llbracket x \downarrow 1 \rrbracket$. We set $\vec{\beta} := (0, 1)$, then compute convolution $\vec{\gamma} := \vec{\alpha} \circledast \vec{\beta} = (0, 0, 0, 2, 1)$. Thm (7d) asserts that

$$\begin{aligned} \mathcal{P}_G(x) &= 2 \cdot \llbracket x \downarrow 3 \rrbracket + \llbracket x \downarrow 4 \rrbracket \\ &= \llbracket x \downarrow 3 \rrbracket [2 + [x - 3]] \stackrel{\text{note}}{=} \llbracket x \downarrow 3 \rrbracket [x - 1]. \end{aligned}$$

This agrees with our derivation via (5a.3). \square

7f: Bipartite ex. Graph $H := \text{Emp}_3$ has $\mathcal{S}(H, 1) = 1$, $\mathcal{S}(H, 2) = 3$, $\mathcal{S}(H, 3) = 1$, and so $\vec{\alpha} = (0, 1, 3, 1)$. Thus

$$\begin{aligned} \mathcal{P}_H(x) &= \llbracket x \downarrow 1 \rrbracket + 3 \llbracket x \downarrow 2 \rrbracket + \llbracket x \downarrow 3 \rrbracket \\ &= x \cdot [1 + [x - 1][3 + [x - 2]]] \\ &= x \cdot [1 + [x^2 - 1]] = x^3. \end{aligned}$$

^{♥4}We can start the sum at $\ell=1$ *except* when G is the void graph. After all, when N is positive then $\mathcal{S}(G, 0)$ is zero.

This indeed agrees with (3a).

Since $K_{3,3}$ is the full-product of H with H , we compute $\vec{\gamma} := \vec{\alpha} \circledast \vec{\alpha} = (0, 0, 1, 6, 11, 6, 1)$. Polynomial

$$\llbracket x \downarrow 2 \rrbracket + 6\llbracket x \downarrow 3 \rrbracket + 11\llbracket x \downarrow 4 \rrbracket + 6\llbracket x \downarrow 5 \rrbracket + \llbracket x \downarrow 6 \rrbracket$$

simplifies (thanks, *Maple*) to

$$x \cdot [x-1] \cdot [x^4 - 8x^3 + 28x^2 - 47x + 31].$$

Happily, this agrees with (6d). \square

Spanning subgraphs

Fix a $G = (\mathbb{V}, \mathbb{E})$ with N vertices and L edges. Each subset $S \subset \mathbb{E}$ can be interpreted as (\mathbb{V}, S) , a spanning subgraph of G . [Thus G has 2^L many spanning subgraphs.] Let $c(S)$ denote the number of connected components of S .

8: CPSS Thm (Chromatic-Poly Spanning Subgraph). *The chromatic polynomial of G satisfies*

$$*: \quad \mathcal{P}_G(x) = \sum_{S: S \subset \mathbb{E}} (-1)^{|S|} \cdot x^{c(S)}. \quad \diamond$$

Exer E4: Prove this. Think Inclusion-Exclusion.
[Hint: See the pamphlet on our TEACHING PAGE.]

Comparison. We can paraphrase Theorems (8) and (7c) as saying: *Spanning subgraphs* express \mathcal{P}_G w.r.t the standard basis $(\mathbf{e}_j)_{j=0}^\infty$ [see (1b)], whereas *Graph-Stirling numbers* write \mathcal{P}_G w.r.t the falling-factorial basis $(\mathbf{t}_n)_{n=0}^\infty$. \square

8a: Ex. As K_3 has 3 edges, it has $2^3 = 8$ spanning subgraphs. The *no-edge* graph has 3 CCs [Connected Components], hence contributes a $[-1]^0 \cdot x^3 = x^3$ term.

The three *one-edge* spanning subgraphs each have 2 CCs; these contribute $3 \cdot [-1]^1 \cdot x^2 = -3x^2$.

The three *two-edge* spanning subgraphs, each have 1 CC, contributing $3 \cdot [-1]^2 \cdot x^1 = 3x$. Finally, the unique *three-edge* spanning subgraph has 1 CC, hence contributing a $[-1]^3 \cdot x^1 = -x$ term. Adding,

$$\begin{aligned} \mathcal{P}_{K_3}(x) &= x^3 - 3x^2 + 3x - x \\ &= x^3 - 3x^2 + 2x \stackrel{\text{note}}{=} \llbracket x \downarrow 3 \rrbracket. \end{aligned} \quad \square$$

Applications/Extensions

Here is one:

9a: Orientations. An **orientation** of $G = (\mathbb{V}, \mathbb{E})$ is putting a direction on each edge, creating a digraph; so G has $2^{|\mathbb{E}|}$ many orientations. An orientation of G is **acyclic** if it has no directed-cycles. Use $\mathcal{A}(G)$ for the number of acyclic orientations of G .

An orientation of K_N is called an **N -tournament**; there are $2^{\binom{N}{2}}$ of them. E.g, K_4 has $2^6 = 64$ orientations. **Exer E5: Prove that $\mathcal{A}(K_N)$ equals $[N!]$.** \square

9b: Acyclic-count Thm (Richard Stanley). *On N -vertex G ,*

$$\mathcal{A}(G) = [-1]^N \cdot \mathcal{P}_G(-1). \quad \diamond$$

Sketch. The idea is to establish an analog of (2a),

$$9c: \quad \mathcal{A}(G) = \mathcal{A}(G \setminus \{\alpha\}) + \mathcal{A}(G/\alpha),$$

for each edge α of G , as follows. Let $M := G \setminus \{\alpha\}$.

- Each acyclic-orient of M extends to *at least one* acyclic-orient of G .
- The number of acyclic-orient of M which give rise to *two* acyclic-orient of G , is $\mathcal{A}(G/\alpha)$. \blacklozenge

10: Generalizing full-product. Let \mathcal{G} be the set of all graphs. Consider a graph $S = (\mathbb{V}, \mathbb{E})$ and a fnc $\mathcal{F}: \mathbb{V} \rightarrow \mathcal{G}$; we'll often write $\mathcal{F}(\mathbf{u})$ as $\mathcal{F}_{\mathbf{u}}$. Use $N_{\mathbf{u}}|L_{\mathbf{u}}$ for the number of vertices|edges of $\mathcal{F}_{\mathbf{u}}$.

The “**full product** of \mathcal{F} over S ”

$$G := \bigodot_S(\mathcal{F})$$

is S , but where each S -vertex \mathbf{u} has been replaced by a copy of graph $\mathcal{F}_{\mathbf{u}}$. Moreover, for each S -edge $\mathbf{v} \text{---} \mathbf{w}$:

Each $\mathcal{F}_{\mathbf{v}}$ -vertex \mathbf{v}' and each $\mathcal{F}_{\mathbf{w}}$ -vertex \mathbf{w}' , are the endpoints of a G -edge.

Thus $\bigodot_S(\mathcal{F})$ has $\sum_{\mathbf{u} \in \mathbb{V}} N_{\mathbf{u}}$ many vertices. It has

$$10a: \quad \left[\sum_{\mathbf{u} \in \mathbb{V}} L_{\mathbf{u}} \right] + \sum_{\mathbf{v}, \mathbf{w}} [N_{\mathbf{v}} \cdot N_{\mathbf{w}}]$$

many edges, where the second sum is over all S -edges $\mathbf{v} \text{---} \mathbf{w}$. In the case that S is a single edge, P_2 , we recover the full-product as defined above (7a).

Full-product from (7a) is associative, so expressions such as $H_1 \odot H_2 \odot \dots \odot H_5$ make sense. If we define $\mathcal{F}(j) := H_j$, then

$$10b: \quad H_1 \odot H_2 \odot \dots \odot H_5 = \bigodot_{K_5}(\mathcal{F}),$$

regarding $[1..5]$ as the vertex-set of K_5 . □

Exer: Is there a formula for the chromatic number/polynomial of $\bigodot_S(\mathcal{F})$, in terms of corresponding information about S and function \mathcal{F} ? What about special cases, e.g $S = P_3$? Or $S = C_{\text{Even}}$?

Filename: `Problems/GraphTheory/chromatic-num.tex`

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