Staple!

Team: ____________________

Abstract Algebra
MAS4301 3175

Home-C

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Hi. Let F and R be the ***flip*** and ***rotation*** in the dihedral group $\mathbb{D}_N$, with $F^2=\mathbf{e}$, $R^N=\mathbf{e}$ and $RFRF=\mathbf{e}$. Use R^j and R^jF as the standard form of each element in $\mathbb{D}_N$.

Use **CN** for *"cycle notation"*, and **DCN** for *"disjoint* **CN**".

C1: Show no work.

____ a) In **CN**, $\alpha^3 = (6\,4\,1\,0\,3\,5\,2) \in \mathbb{S}_7$. So $\alpha=$ ____________.

____ b) Perm $\beta \in \mathbb{S}_{15}$ has sig $\lceil 5^3 \rfloor$. It has ____ many sqroots with sig $\lceil 5^3 \rfloor$, and ________________ with sig $\lceil 10^1, 5^1 \rfloor$.

____ c) Let $V_K := G \times G \times \overset{K}{\cdots} \times G$, where $G := (\mathbb{Z}_2, +)$. Define $A_K := \operatorname{Aut}(V_K)$. Then $A_2 \cong \mathbb{D}_N$, where $N=$ ____.
As a decreasing product of integers,
$|A_3|=$ ____________ and $|A_4|=$ ____________________.

____ d) $G := \mathbb{Z}_{27} \times \mathbb{Z}_9$ has ________________ *elements* of order-9. So G has ____________ *cyclic subgps* of order-9.

____ e) The OP-isometry group, G, of the tetrahedron, $\mathbf{T}$, has $|G|=$ ______. K-coloring the edges of $\mathbf{T}$, there are
____ $\cdot\,[K^6 +$ ______________________ $]$ G-distinct colorings.
[Burnside's thm, P.474.]

Essay question: **Fill-in all blanks.**

C2: The 4×4 TTT (TicTacToe) board is $\mathbb{B} := [1..4] \times [1..4]$. Let Γ denote the TTT-automorphism group; the set of self-bijections of $\mathbb{B}$ which preserve all ten TTTs. So $R, F \in \Gamma$, where R rotates $\mathbb{B}$ by 90° CCW, and F flips $\mathbb{B}$ about its vertical axis. Evidently, $\langle R, F \rangle_\Gamma \cong \mathbb{D}_4$.

A less obvious TTT-aut is the ***swizzle***, S: It exchanges each corner square with the central square that it (diagonally) touches; and it does The Right Thing on the edge squares.

i) Easily, $S \leftrightarrows R$ and $S \leftrightarrows F$, hence each element of subgroup $\Lambda := \langle S, F, R \rangle$ can be written in form $S^b F^c R^d$ for integers b, c, d. So $|\Lambda|=$ ______; prove this.

Draw a "16-dot picture" (4×4 dots, with arrows), for each element $\alpha \in \Lambda$. However, use the same picture for α rotated or flipped about any line, or for α^{-1} (rotated or flipped). *Label* each picture with all the automorphisms that it describes. The total number of labels should equal your $|\Lambda|$.

ii) Find a TTT-aut T which is **not** in the Λ subgp. Write the commutation relations between T and each of $\{S, F, R\}$. Prove that each aut α can be written as $\alpha = T^{a_1} S^{a_2} F^{a_3} R^{a_4}$, with $a_i \in \mathbb{Z}$. For element $\beta = T^{b_1} S^{b_2} F^{b_3} R^{b_4}$, give an explicit multiplication rule showing how to compute the exponents $\{c_i\}_{i=1}^4$ of $\beta\alpha = T^{c_1} S^{c_2} F^{c_3} R^{c_4}$.

Prove that $\langle T, S, F, R \rangle$ is **all** of Γ. Thus $|\Gamma|=$ ______.

Draw all the new labeled 16-dot pictures for $\Gamma \smallsetminus \Lambda$.

iii) Find a set of *involutions* which generates Γ. Compute (with proof) the center of Γ; what is its order?

iv) With $\mathbf{u} \in \mathbb{B}$ the upper-LH corner of $\mathbb{B}$, define its stabilizer $\Upsilon := \operatorname{Stab}_\Gamma(\mathbf{u})$. With proof, compute $|\Upsilon|=$ ____.
The number of Υ-orbits is ____. (This is one more than the number of *"really different"* replies to a corner first-move.)

v) On the 4×4×4 TicTacToe board (Qubic), what is a 3-dimensional analog of The Swizzle? How many (with proof) *really different* first moves are there? (ExtraCredit: Give, with proof, a generating set for the group TTT-autos, Γ_{Qubic}.)

End of Home-C

C1: ___ ___ ___ 150pts

C2: ___ ___ ___ 125pts

Total: ___ ___ ___ 275pts

HONOR CODE: *"I have neither requested nor received help on this exam other than from my team-mates and my professor (or his colleague)."* *Name/Signature/Ord*

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