ACES
MAA4102, MAA5104

Prac-C

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Hello. Please write **DNE** in a blank <u>if</u> the described object does not exist or if the indicated operation cannot be performed.

Fill-in *all* blanks on this sheet **<u>including</u>** the blanks for the essay questions!

C0: Show no work.

a) Repeating decimal $0.7\overline{20}$ equals $\frac{n}{d}$, where posints $n \perp d$ are $n=$ ________ and $d=$ ________.

b) MS (X, d) is ***connected*** IFF (Defn.)

____
________.

c) MS (X, d) is (***metrically***) ***complete*** IFF (Defn.)

________.

d) THM: $\forall A, B \subset X$, if ________

then [Circle *relation* and circle *binary operator*]

$\text{Diam}(A) + \text{Diam}(B) \quad \leqslant \; = \; \geqslant \quad \text{Diam}(A \cup \cap B)$.

e) GS-THM (13Oct.): Fix a normed VS $(W, \|\cdot\|)$ and seq $\vec{\mathbf{p}} \subset W$. Suppose $\exists$ ________

st. $\forall n$: $\|p_{n+1} - p_n\|$ ________.

Then $\vec{\mathbf{p}}$ is $\|\cdot\|$-Cauchy.

f) TALL-MAN THM (13Oct.): Fix a totally-ordered space $(T, \leqslant)$. Then each sequence $\vec{\mathbf{x}} \subset T$ ________

____
____

g) A map $f\colon \mathbf{V} \times \mathbf{E} \to \mathbf{W}$ (where $\mathbf{V}, \mathbf{E}, \mathbf{W}$ are $\mathbb{R}$-vectorspaces) is ***bilinear*** if $\forall \mathbf{v}_1, \mathbf{v}_2 \in \mathbf{V}$, $\forall \mathbf{e}_1, \mathbf{e}_2 \in \mathbf{E}$ and

$\forall$ ________ : ________

and ________.

A map $\langle \cdot, \cdot \rangle$ from $\mathbf{V} \times \mathbf{V} \to \mathbb{R}$ is an ***inner product*** if

I1: ________.

I2: ________.

I3: ________.

h) On $\mathbb{R}$-VS X, a ***norm*** $\|\cdot\|$ is a map ______ $\to$ ______ satisfying these three axioms. [*Hint:* Quantifiers.]

N1: ________.

N2: ________.

N3: ________.

N4: ________.

i) Let $\mathbf{v} := (2, 3) \in \mathbb{R}^2$; so $\|\mathbf{v}\|_3 =$ ________.

j) Let E be the set $\left\{5 + \left[[-1]^n \cdot \frac{3n-1}{n}\right]\right\}_{n \in \mathbb{Z}_+}$. Then $\sup_{\mathbb{R}}(E) =$ ________ and $\inf_{\mathbb{R}}(E) =$ ________.

k) On a set Y, a ***metric*** m is a map

______ $\to$ ______ such that $\forall$ ________ :

MS1: ________.

MS2: ________.

MS3: ________.

MS4: ________.

l) With $\boldsymbol{\alpha}(\cdot, \cdot)$ the arctan metric on $\mathbb{R}$, the $\boldsymbol{\alpha}$-Diam(PRIMES)= ________.

[*Hint:* No $\boldsymbol{\alpha}()$ should appear in your ans. But arctan() can.]

m) $\mathcal{P}(\mathcal{P}(\text{3-stooges}))$ has ________ many elements.

C1: Show no work.

a) Suppose that $U, V_1, V_2, \ldots$ are $\mathbb{R}$-open-sets, and $E, K_1, K_2, \ldots$ are $\mathbb{R}$-closed-sets. [Circle] those of the following sets which are <u>guaranteed</u> to be $\mathbb{R}$-*closed.*

$K_1 \smallsetminus E$. $\quad \partial_{\mathbb{R}}(E) \cap \text{Itr}_{\mathbb{R}}(E)$. $\quad \partial_{\mathbb{R}}(E) \cup \text{Itr}_{\mathbb{R}}(E)$.

$\mathbb{R} \smallsetminus \left[\bigcup_{n=1}^{\infty} V_n\right]$. $\quad \bigcup_{n=1}^{\infty} \text{Cl}_{\mathbb{R}}(V_n)$. $\quad [\text{Itr}_{\mathbb{R}}(E) \cap V_1]^c$.

b) Define $X :=$ ________ $\subset \mathbb{R}$ st. the X-open ball $B := X\text{-Bal}_3(0) =$ ________ satisfies $B \subsetneqq \text{Cl}_X(B) =$ ________ $\subsetneqq X\text{-CldBal}_3(0) =$ ________

c) Sets $A :=$ ________ and $B :=$ ________ have $\partial_{\mathbb{R}}(A) =$ ________ and $\partial_{\mathbb{R}}(B) =$ ________. Moreover,

______ $= \partial_{\mathbb{R}}(A) \cap \partial_{\mathbb{R}}(B) \subsetneqq \partial_{\mathbb{R}}(A \cap B) =$ ______.

d Sets $C :=$ ______ and $D :=$ ______ have $\partial_{\mathbb{R}}(C) =$ ______ and $\partial_{\mathbb{R}}(D) =$ ______. Further,

______ $= \partial_{\mathbb{R}}(C) \cap \partial_{\mathbb{R}}(D) \supsetneqq \partial_{\mathbb{R}}(C \cap D) =$ ______.

e In $\mathbb{R}$, open intervals $J_n := ($ ______ , ______ $)$ intersect to a **non**-open set $\bigcap_{n=1}^{\infty} J_n =$ ______.

f Let $S := \{q \in \mathbb{Q}_+ \mid 5 \leqslant q^2 < 9\}$. Then:

$\mathrm{Cl}_{\mathbb{R}}(S) =$ ______. $\mathrm{Itr}_{\mathbb{R}}(S) =$ ______.

$\mathrm{Cl}_{\mathbb{Q}}(S) =$ ______. $\mathrm{Itr}_{\mathbb{Q}}(S) =$ ______.

g Let $S := \{q \in \mathbb{Q}_+ \mid 5 \leqslant q^2 < 9\}$. Then:

$\partial_{\mathbb{R}}(S) =$ ______. $\partial_{\mathbb{Q}}(S) =$ ______.

C2: Show no work.

a Let $\mathbf{v} := (3, -3, 2, 1, 1) \in \mathbb{R}^5$; so $\|\mathbf{v}\|_3 =$ ______.

b Let $f : \mathbb{R} \to \mathbb{R}$ by $f(x) := [x-1]^2 - 3$. Define restrictions $g := f\lfloor_{[-1,2]}$ and $h := f\lfloor_{[-2,2]}$. Then the sup-norm $\|g\|_{\sup} =$ ______ and $\|h\|_{\sup} =$ ______.

c Using the stereographic-metric on $\dot{\mathbb{R}}$:

$\boldsymbol{\sigma}(-1, 0) =$ ______, $\boldsymbol{\sigma}(\infty, 0) =$ ______. For $n \in \mathbb{Z}_+$, distance $\boldsymbol{\sigma}(-n, n) =$ ______. [*Hint:* OTForm Formula(n).]

d Under stereographic-projection the point $3 \in \dot{\mathbb{R}}$ maps to (x, y), where $x =$ ______ and $y =$ ______.

So $[\boldsymbol{\sigma}(\infty, 3)]^2 =$ ______.

Essay questions: **Fill-in all blanks.** *For each question, carefully write a triple–spaced essay solving the problem.*

C3: Define: "On a set E, a binary relation ∇ is an ***equivalence relation*** IFF...". *Make sure* to define any terms like "reflexive" that you use in your defn.!

Let $\mathbf{P}$ be the set of ordered integer-pairs (n, d), with $d \neq 0$. Define relation \$ on $\mathbf{P}$ by

$$(N, D) \ \$ \ (x, y) \qquad \text{IFF} \qquad N \cdot y = x \cdot D.$$

Prove, in detail, that \$ is an equivalence relation.

C4: Sets U_i are open in MS (Ω, d). i Prove that $U_1 \cup U_2$ is Ω-open.

ii Prove that $U_1 \cap U_2$ is Ω-open.

C5: Use $\langle \cdot, \cdot \rangle$ for an inner-product on $\mathbb{R}$-vectorspace $\mathbf{V}$.

x1 State the Cauchy-Schwarz Inequality Thm, carefully stating the IFF-condition for equality. x2 **Prove** the C-S Inequality Thm, using the axioms for inner-product.

C6: State and prove the Monotone-subsequence thm.

C7: State and prove the Intermediate-value theorem.

C8: We have seqs $\vec{\mathbf{x}}, \vec{\mathbf{y}} \subset \mathbb{R}$ with $\lim(\vec{\mathbf{x}}) = 2$ and $\lim(\vec{\mathbf{y}}) = 4$. Letting $p_n := x_n \cdot y_n$, give a rigorous ε-proof that $\lim(\vec{\mathbf{p}}) = 8$. (Do not restate the problem. You may quote, without proof, a theorem about the diameter of convergent sequences.)

C9: In a normed-VS $(\mathbf{W}, \|\cdot\|)$, suppose we have a sequence $\vec{\mathbf{x}} \in \mathbf{W}$ and a number $r \in [0, 1)$ such that

$\forall n \in \mathbb{Z}_+ : \|x_n - x_{n+1}\| \leqslant r^n$. Prove that sequence $\vec{\mathbf{x}}$ is $\|\cdot\|$-Cauchy.